Unit - VI

Stability Analysis

Introduction

Mapping Between The s plane and the z plane

Mapping of left half of splane into the z plane

In the design of continuous time control systems, the location of poles and zeros in the splane are very important in predicting the dynamic behaviour of the system. Similarly in designing discrete time control systems the location of poles and zeros in the z plane are very important.

When impulse ~~signal~~ sampling is incorporated into the process, the complex variables z and s are related by the equation

$$z = e^{Ts}$$

This means a pole in the s plane can be located in the z plane through the transformation of $z = e^{Ts}$.

Since the complex variable s has real part σ and imaginary part ω, we have

$$s = \sigma + j\omega$$

and $$z = e^{T(\sigma + j\omega)} = e^{T\sigma} \cdot e^{jT\omega} = e^{T\sigma} \cdot e^{j(T\omega + 2\pi k)}$$

We see that poles and zeros in the splane, where frequencies differ in integral multiples of the sampling frequency $\frac{2\pi}{T}$ are ~~mapt~~ mapped into the same locations in the z plane. This means that there are infinitely many values of s for each value of z.

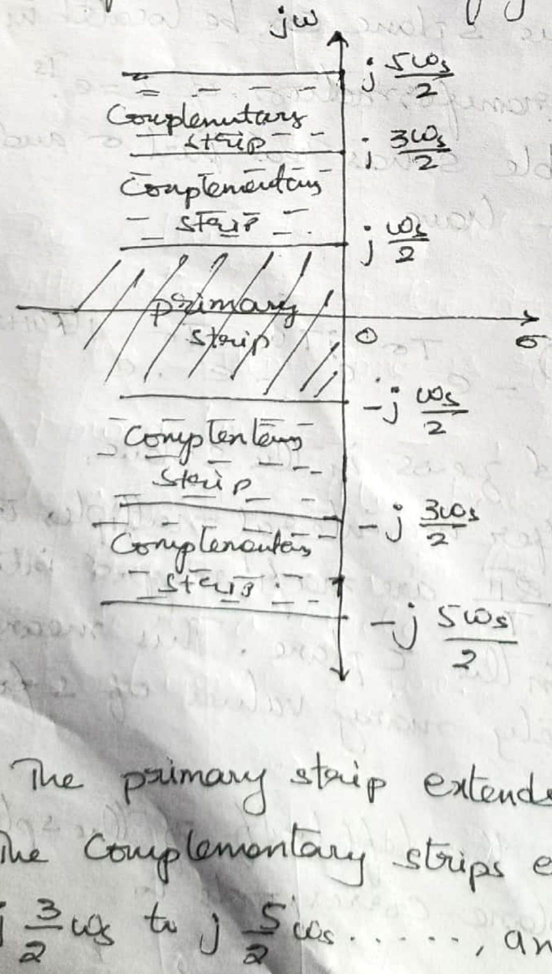
Since σ is negative in the left half of the splane, the left half of the splane corresponds to

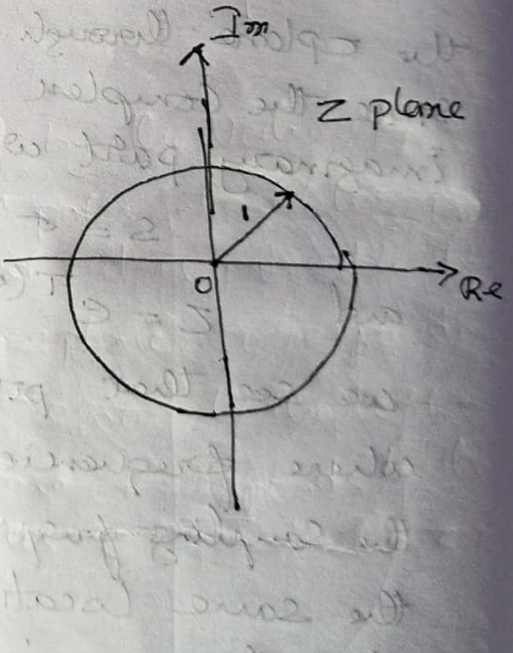
$$|z| = e^{T\sigma} < 1$$

The $j\omega$ axis in the s plane corresponds to $|z| = 1$. That is the imaginary axis in the splane (line $\sigma = 0$) corresponds to the unit circle in the z plane, and the interior of unit circle corresponds to the left half of s plane

primary strip and Complementary strips

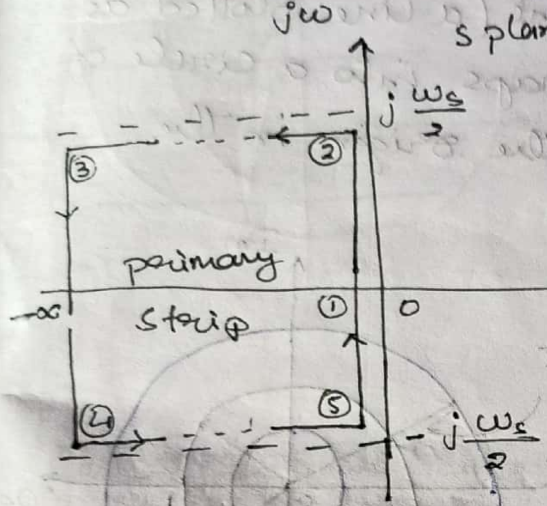
The angle $\angle z = \omega T$ of z varies from $-\infty$ to ∞ as ω varies from $-\infty$ to ∞. Consider a representative point on the $j\omega$ axis in the s plane. As this point moves from $-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$ on the $j\omega$ axis, where ω_s is the sampling frequency, we have $|z| = 1$ and $\angle z$ varies from $-\pi$ to π in the counterclockwise direction in the z plane. As the point moves from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$ on the $j\omega$ axis, the corresponding point in the z plane traces out the unit circle once in the counterclockwise direction. Thus, as the point in the s plane moves from $-\infty$ to ∞ on the $j\omega$ axis, we trace the unit circle in the z plane an infinite number of times. It is clear that each strip of width ω_s in the Left half of the s plane maps into the inside of the unit circle in the z plane. This implies that left half of the s plane may be divided into an infinite number of periodic strips as shown in figure

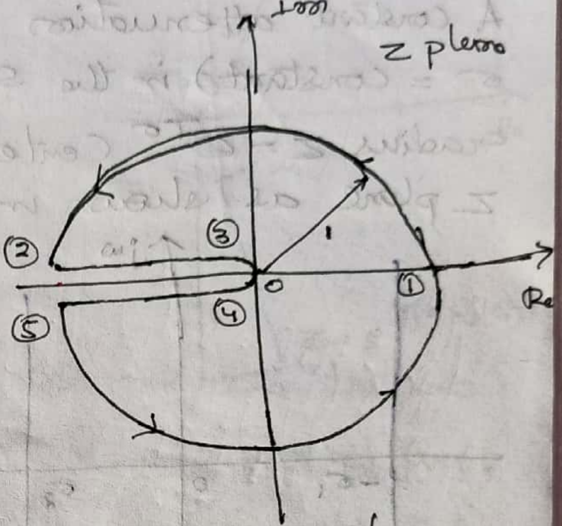




The primary strip extends from $j\omega = -j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$. The Complementary strips extend from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$, $j\frac{3}{2}\omega_s$ to $j\frac{5}{2}\omega_s$ . . . . . , and from $-j\frac{1}{2}\omega_s$ to $-j\frac{3}{2}\omega_s$, $-j\frac{3}{2}\omega_s$ to $-j\frac{5}{2}\omega_s$ . . . . .

In the primary strip, if we trace the sequence of points 1-2-3-4-5-1 in the s plane as shown by the circled numbers in figure, then this path is mapped into the unit circle centered at the origin of the z plane as shown in figure. The corresponding points 1, 2, 3, 4 & 5 in the z plane are shown by the circled numbers in figure





The area enclosed by any of the complementary strips is mapped into the same unit circle in the z plane. This means that the correspondence between the z plane and the s plane is not unique. A point in the z plane corresponds to an infinite number of points in the s plane, although a point in the s plane corresponds to a single point in the z plane

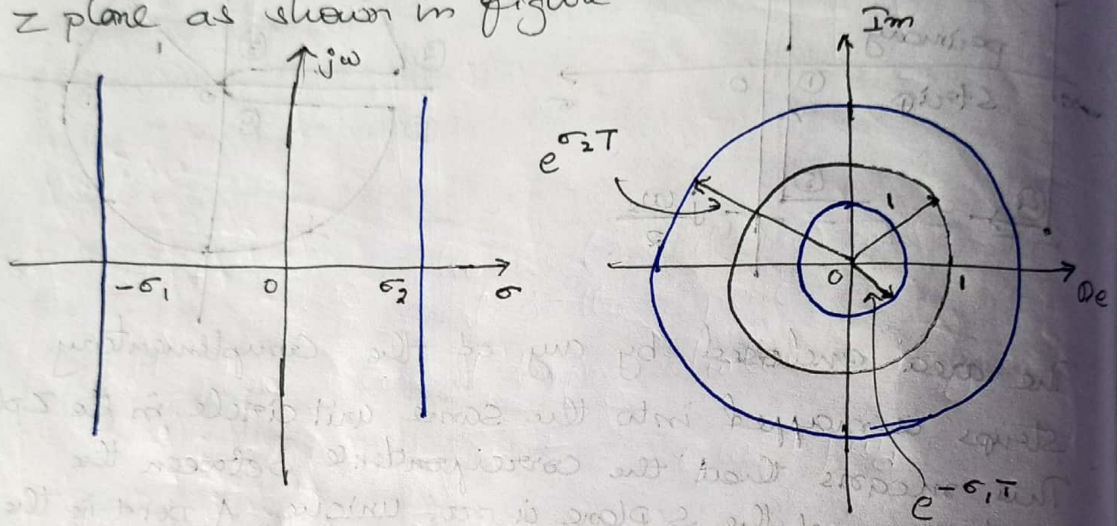
Since the entire left half of the s plane is mapped into the interior of the unit circle in the z plane, the entire right half of the s plane is mapped into the exterior of the unit circle in z plane. the imaginary jω axis in the s plane maps into the unit circle in the z plane. if the sampling frequency is at least twice as fast as the highest frequency component involved in the system, then every point in the unit circle in the z plane represents frequencies between $-\frac{1}{2}\omega_s$ to $\frac{1}{2}\omega_s$.

Commonly used Contours in the s plane into z plane

Some commonly used contours in the s plane is attenuation loci, constant frequency loci and Constant damping ratio loci.

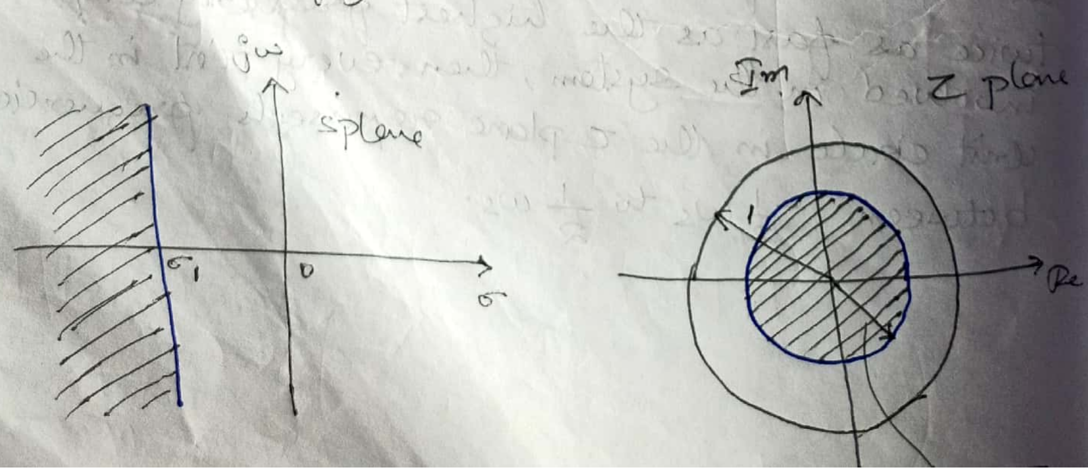
Constant Attenuation Loci

A constant attenuation loci line (a line plotted as σ = constant) in the s plane maps into a circle of radius $z = e^{T\sigma}$ centered at the origin in the z plane as shown in figure



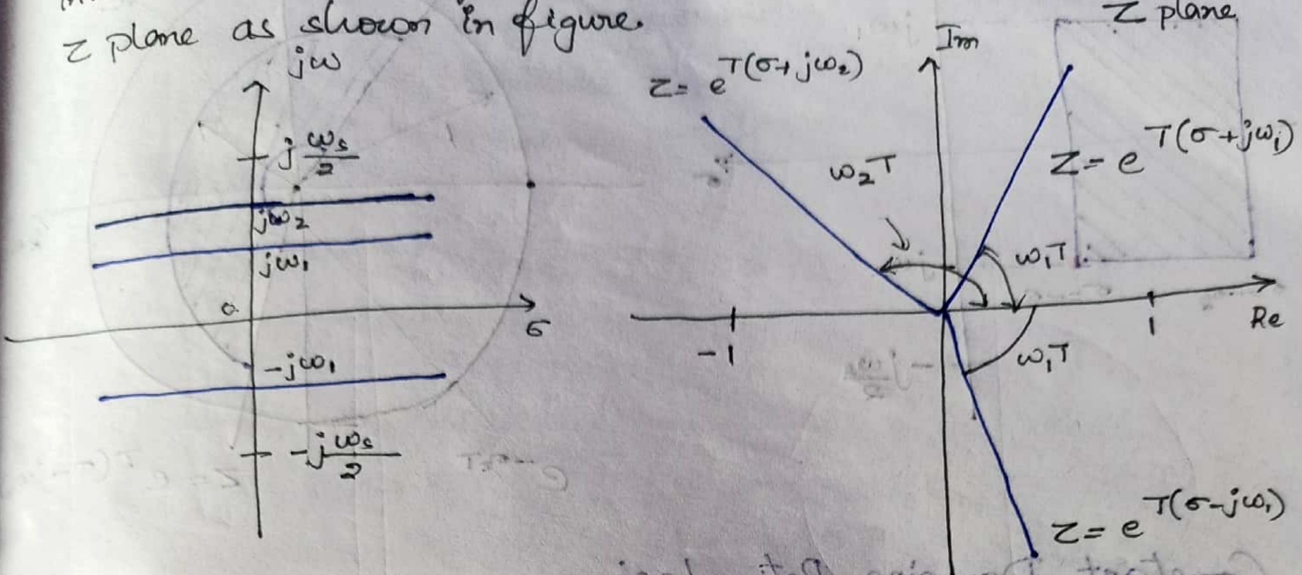
Settling time t_s

The settling time is determined by the value of attenuation σ of the dominant ~~poles~~ closed loop poles. if the settling time is specified, it is possible to draw a line $\sigma = -\sigma_1$ in the s plane corresponding to a given settling time. the region to the left of the line $\sigma = -\sigma_1$ in the s plane corresponds to the inside of a circle with radius $e^{-\sigma_1 T}$ in the z plane as shown in figure



Constant Frequency Loci

A constant frequency locus $\omega = \omega_1$ in the s plane is mapped into a radial line of constant angle $T\omega_1$ (in radians) in the z plane as shown in figure.



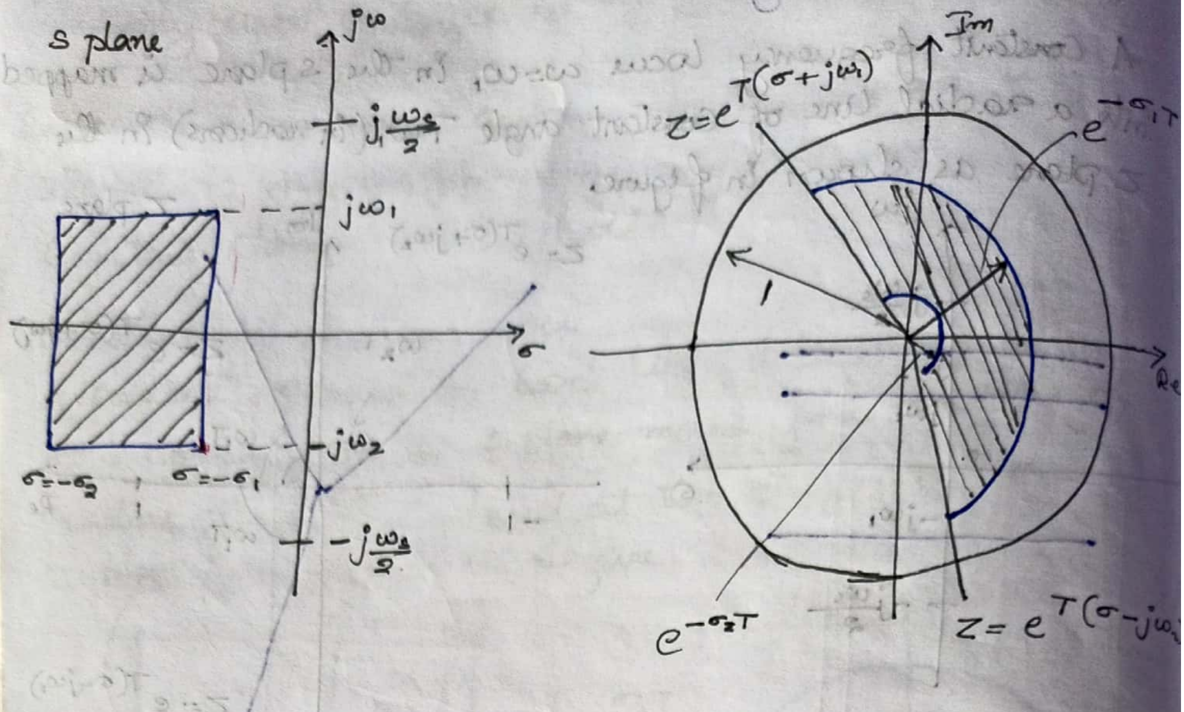
Constant-frequency lines at $\omega = \pm\frac{1}{2}\omega_s$, in the left half of the s plane correspond to the negative ~~real~~ real axis in the z plane between 0 and -1. Since $T(\pm\frac{1}{2}\omega_s) = \pm\pi$.

Constant frequency lines at $\omega = \pm\frac{1}{2}\omega_s$ in the right half of the s plane correspond to the negative real axis in the z plane between -1 and $-\infty$.

The negative real axis in the s plane corresponds to the positive real axis in the z plane between 0 and 1.

Constant frequency lines at $\omega = \pm n\omega_s$ $(n = 0, 1, 2 \ldots)$ in the right half of the s plane map into the positive real axis in the z plane between 1 and ∞.

The region bounded by constant frequency lines $\omega = \omega_1$ and $\omega = -\omega_2$ (where ω_1 & ω_2 lie between $-\frac{1}{2}\omega_s$ and $\frac{1}{2}\omega_s$) and constant attenuation lines $\sigma = -\sigma_1$ and $\sigma = -\sigma_2$ as shown in figure is mapped into a region bounded by two radial lines and two circular arcs as shown in figure.



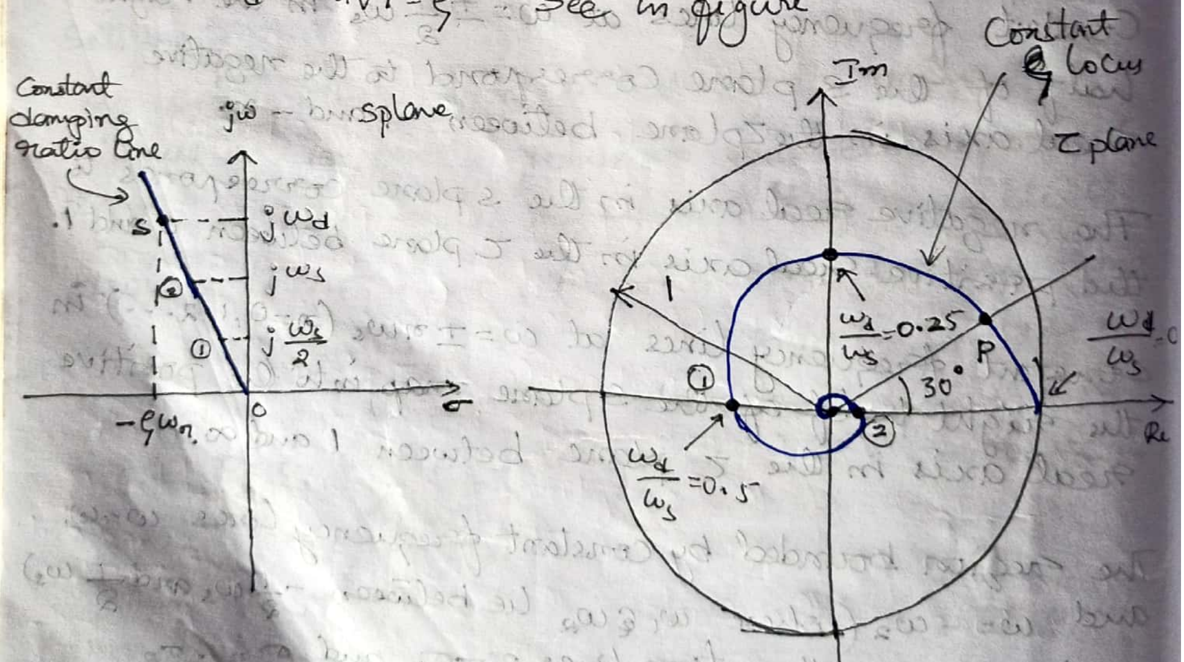
Constant Damping Ratio Loci

A Constant damping ratio line (radial line) in the s plane is mapped into a spiral in the z plane.

In the s plane, a constant damping ratio line can be given by

$$s=-\xi\omega_n+j\omega_n\sqrt{1-\xi^2}=-\xi\omega_n+j\omega_d$$

where $\omega_d=\omega_n\sqrt{1-\xi^2}$ See in figure



In the z plane this line becomes

$$z=e^{Ts}=\exp\left(-\xi\omega_n T+j\omega_d T\right)$$

$$=\exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s}+j2\pi\frac{\omega_d}{\omega_s}\right)$$

hence $|z| = \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s}\right)$ and

$\angle z = 2\pi \frac{\omega_d}{\omega_s}$

The magnitude of z decreases and the angle of z increases linearly as ω_d increases and the locus in the z plane becomes a Logarithmic spiral as shown in figure.

Stability Analysis of Closed loop systems in the z plane

Stability of linear time invariant single input single output discrete time control system.

consider closed loop pulse transfer function system

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \quad \text{——— (1)}$$

The stability of the system defined by above equation as well as of other types of discrete time control systems, may be determined from the locations of the closed loop poles in the z plane or the roots of the characteristic equation

$$P(z) = 1 + GH(z) = 0$$ as follows

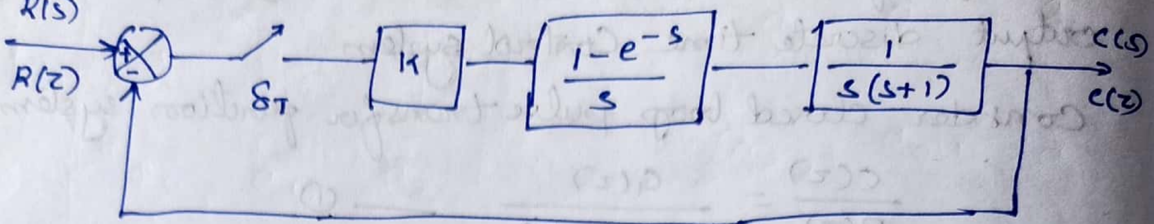
1. For the system to be stable, the closed loop poles or the roots of the characteristic equation must lie within the unit circle in the z plane. Any closed loop pole outside the unit circle makes the system unstable.

2. If a simple pole lies at $z=1$, then the system becomes critically stable. also, the system becomes the ~~system~~ critically stable if a single pair of conjugate complex poles lies on the unit circle in the z plane. Any multiple closed loop poles on the unit circle makes the system unstable

3. closed loop zeros do not affect the absolute stability and therefore may be located any where in the z plane.

A linear time invariant single input single output discrete time closed loop control system becomes unstable if any of the closed loop poles lies outside the unit circle and or any multiple closed loop pole lies on the unit circle in the z plane.

* Consider the closed loop control system shown in figure. Determine the stability of the system when $k=1$



Sol: The open loop transfer function G(s) of the system is

$$G(s) = 1 \cdot \frac{1-e^{-s}}{s} \cdot \frac{1}{s(s+1)}$$

The Z transform of G(s) i.e.,

$$G(z) = Z\left[\frac{1-e^{-s}}{s^2(s+1)}\right] = \frac{0.3679z + 0.2642}{(z-0.3679)(z-1)}$$

closed loop pulse transfer function for the system

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)}$$

characteristic equation $1 + G(z) = 0$

$$(z-0.3679)(z-1) + 0.3679z + 0.2642 = 0$$

$$z^2 - z + 0.6321 = 0$$

$$z_1, z_2 = 0.5 \pm j0.6181$$

since $|z| = |z_2| < 1$, system is stable

In the absence of sampler a second order system is always stable. In the presence of sampler a second order system can becomes unstable for large values of gain.

Three stability tests can be applied directly to the characteristic equation $P(z)=0$ without solving the roots

1. Schur - cohn stability test
2. Jury stability test
3. Bilinear transformation coupled with Routh stability criterion

The first two tests reveal the existence of any unstable roots (roots lie outside the unit circle in z plane). however these tests neither give the locations of unstable roots nor indicate the effects of parameter changes on the system stability except for the simple case of low order systems.

~~The third method is based~~

Both the schur - cohn stability test and the Jury stability test may be applied to polynomial equations with real or complex coefficients when The computations required in Jury test, involves only real the polynomial equation coefficients

Jury stability Test

In applying the Jury stability test to a given characteristic equation $P(z)=0$, we construct a table whose elements are based on the coefficients of $P(z)$.

Assume that the characteristic equation $P(z)$ is a polynomial in z as follows

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

where $a_0 > 0$. Then the Jury table becomes

Row	z^0	z^1	z^2	z^3	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	b_3	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	$\cdots$	c_0		
6	c_0	c_1	c_2	c_3	$\cdots$	c_{n-2}		
$\vdots$	$\vdots$							
$2n-5$	p_3	p_2	p_1	p_0				
$2n-4$	p_0	p_1	p_2	p_3				
$2n-3$	q_2	q_1	q_0					

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}, \quad k = 0, 1, 2, \ldots, n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}, \quad k = 0, 1, 2, \ldots, n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}, \quad k = 0, 1, 2$$

The last row in the table consists of three elements (For second order systems, $2n-3 = 1$ and the Jury table consists only one row containing three elements)

Notice that the elements in any even numbered row are simply the reverse of the immediately preceding odd numbered row.

Stability Criterion by the Jury test

A stability with the characteristic equation $P(z)=0$ given by equation

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

where $a_0 > 0$, is stable if the following conditions are all satisfied.

1. $|a_n| < a_0$
2. $P(z)\big|_{z=1} > 0$
3. $P(z)\big|_{z=-1} \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd} \end{cases}$
4. $|b_{n-1}| > |b_0|$

   $|c_{n-2}| > |c_0|$

   $\vdots$

   $|q_2| > |q_0|$

* Construct the Jury stability table for the following characteristic equation

$$P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

where $a_0 > 0$. Write the stability conditions

Sol: The stability conditions are as follows

1. $|a_4| < a_0$
2. $P(1) = a_0 + a_1 + a_2 + a_3 + a_4 > 0$ $\quad n = 4 = \text{even}$
3. $P(-1) = a_0 - a_1 + a_2 - a_3 + a_4 > 0$
4. $|b_3| > |b_0|$, $|c_2| > |c_0|$

Referring to the general case of the Jury stability table, a Jury stability table for the fourth order system may be constructed as shown in table. The table is slightly modified from the standard form and is convenient for the computations of b's and c's values. The determinant given in middle of each row gives the value of b or c written on the right hand side of same row.

Row	z^0	z^1	z^2	z^3	z^4	
	$\begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix}$					$= b_3$
	$\begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix}$					$= b_2$
	$\begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix}$					$= b_1$
1 2	$\begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix}$					$= b_0$
	$\begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix}$					$= c_2$
	$\begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix}$					$= c_1$
3 4	$\begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}$					$= c_0$
5	c_2	c_1	c_0			

Note that value of c_1 is not used in the stability test and therefore the computation of c_1 may be omitted.

* Examine the stability of the following characteristic equation

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

Sol: For characteristic equation

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

$a_0 = 1$ $a_1 = -1.2$ $a_2 = 0.07$ $a_3 = 0.3$ $a_4 = -0.08$

1. $|a_4| < a_0$ $\therefore$ clearly first condition is satisfied

2. $P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08$
   $= 0.09 > 0$
   Second condition is satisfied

3. $P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08$ $n = 4 =$ even
   $= 1.89 > 0$
   Third condition is satisfied.

Row	z^0	z^1	z^2	z^3	z^4
1	-0.08	0.3	0.07	-1.2	1
2	1	-1.2	0.07	0.3	-0.08
3	-0.994	1.176	-0.0756	-0.204	
4	-0.204	-0.075	1.176	-0.994	
5	0.946	-1.184	0.315		

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$$b_0 = \begin{vmatrix} -0.08 & 0.3 \\ 1 & -1.2 \end{vmatrix} = -0.204$$

$$c_2 = \begin{vmatrix} -0.994 & -0.204 \\ -0.204 & -0.994 \end{vmatrix} = 0.946$$

$$c_0 = \begin{vmatrix} -0.994 & 1.176 \\ -0.204 & -0.0756 \end{vmatrix} = 0.315$$

$$|b_3| > |b_0| \; , \; |c_2| > |c_0|$$

Thus both the fourth conditions is satisfied since all conditions for stability are satisfied the given characteristic equation is stable., or all roots lie inside the unit circle in the z plane.

~~As~~ The given characteristic equation $P(z)$ can be factored as

$$P(z) = (z-0.8)(z+0.5)(z-0.5)(z-0.4)$$

all the roots are with in the unit circle in z plane.

* Examine the stability of the characteristic equation given by

$$P(z) = z^3 - 1.1z^2 - 0.1z + 0.2 = 0$$

Sol: coefficients $a_0 = 1$ $a_1 = -1.1$ $a_2 = -0.1$ $a_3 = 0.2$

The conditions for stability in the Jury test for the third order system

1. $|a_3| < a_0$

2. $P(1) > 0$

3. $P(-1) < 0$ $\quad n = 3 = \text{odd}$

4. $|b_2| > |b_0|$

1. The first condition $|a_3| < a_0$ is satisfied.

$P(1) = 1 - 1.1 - 0.1 + 0.2 = 0$

This indicates, at least one root is at $z = 1$.
Therefore system is best critically stable.
The remaining tests determine whether the system is critically stable or unstable
(if the given characteristic equation represents a control system, critically stability will not be desired. The stability test may be stopped at this point)

The third condition is satisfied, we fourth condition of the jury test

$b_0 = -0.12$

$b_2 = -0.96$

$|b_2| > |b_0|$

fourth condition is satisfied.

from the above analysis we conclude that given characteristic equation has one root on the unit circle $(z = 1)$ and its other two roots with in the unit circle in the z plane
system is critically stable.

* A control system has the following characteristic equations

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$$

Determine the stability of the system

Sol: $a_0 = 1$, $a_1 = -1.3$, $a_2 = -0.08$, $a_3 = 0.24$

first condition for stability $|a_3| < a_0$ is satisfied.

Second condition

$$P(1) = 1 - 1.3 - 0.08 + 0.24 = -0.14 < 0$$

The system is unstable.

* Consider the discrete time unity feedback control system (sampling period $T = 1$ sec) whose open loop transfer function is given by

$$G(z) = \frac{K(0.3679z + 0.2642)}{(z - 0.3679)(z - 1)}$$

Determine the range of gain K for stability by use of Jury stability test.

Sol: The closed loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{k(0.3679z + 0.2642)}{z^2 + (0.3679k - 1.3679)z + 0.3679 + 0.2642k}$$

Thus C.E of the system

$$P(z) = z^2 + (0.3679K - 1.3679)z + 0.3679 + 0.2642K = 0$$

second order system Jury stability conditions may be written as

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0$, $n = 2 = $ even

First condition for stability since

$a_2 = 0.3679 + 0.2642K$ and $a_0 = 1$, the first condition for stability becomes

$$|0.3679 + 0.2642K| < 1$$

or $\quad 2.3925 > K > -5.1775$

second condition for stability becomes

$$P(1) = 1 + (0.3679K - 1.3679) + 0.3679 + 0.2642K$$

$$= 0.6321K > 0$$

which gives $K > 0$

Third condition for stability becomes

$$P(-1) = 1 - (0.3679K - 1.3679) + 0.3679 + 0.2642K$$ which yields

$$= 2.7358 - 0.1037K > 0$$

$$26.3822 > K$$

For stability, gain constant K must satisfy inequalities in above analysis hence

$$2.3925 > K > 0$$

The range of gain constant K for stability is between 0 and 2.3925.

If gain K is set equal to 2.3925, system becomes critically stable $K = 2.3925$

C.E becomes $\quad z^2 - 0.4877z + 1 = 0$

$$z = 0.2439 \pm j0.9698$$

Sampling period $T = 1$ sec $\quad \angle z = \tan^{-1}\left(\frac{0.9698}{0.2439}\right)$

$\angle z = 2\pi \frac{\omega_d}{\omega_s} \Rightarrow \omega_d = \frac{\omega_s}{2\pi} \angle z = \frac{2\pi}{2\pi} \angle z = 1.3244$ rad/sec

frequency of sustained oscillations.

Stability Analysis by use of Bilinear Transformation and Routh stability Criterion

Third method used in stability analysis of discrete time control systems is to use the bilinear transformation Coupled with the Routh stability criterion. This method requires transformation from the z plane to another Complex plane, the w plane

The bilinear transformation defined by

$$z = \frac{w+1}{w-1}$$ which solved for w

$$w = \frac{z+1}{z-1}$$

maps the inside of the unit circle in the z plane into the left half of the w plane.

Let the real part of w be called σ and the imaginary part ω so that

$$w = \sigma + j\omega$$

Since the inside of the unit circle in the z plane is

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\sigma + j\omega + 1}{\sigma + j\omega - 1}\right| < 1$$

$$\frac{(\sigma+1)^2 + \omega^2}{(\sigma-1)^2 + \omega^2} < 1$$

we get $(\sigma+1)^2 + \omega^2 < (\sigma-1)^2 + \omega^2$

which yields $\sigma < 0$

Thus, the inside of the unit circle in the z plane ($|z|<1$) corresponds to the left half of the w plane. The unit circle in the z plane is mapped into the imaginary axis in the w plane, and the outside of the unit circle in the z plane is mapped into the right half of the w plane.

In stability analysis using the bilinear transformation coupled with the Routh stability criterion. Substitute $\frac{w+1}{w-1}$ for z in the characteristic equation

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$$

$$\therefore a_0\left(\frac{w+1}{w-1}\right)^n + a_1\left(\frac{w+1}{w-1}\right)^{n-1} + a_2\left(\frac{w+1}{w-1}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{w+1}{w-1}\right) + a_n = 0$$

multiply $(w-1)^n$ to the total equation we obtain

$$Q(w) = b_0 w^n + b_1 w^{n-1} + \cdots + b_{n-1} w + b_n = 0$$

once we transform $P(z)=0$ into $Q(w)=0$, it is possible to apply Routh stability criterion in the same manner as in continuous time systems.

Note that the bilinear transformation coupled with Routh stability criterion will indicate exactly how many roots of the characteristic equation lie in the right half of the w plane and how many lie on the imaginary axis

The amount of computation required in this approach is much more than that required in the Jury stability test.

* Consider the following characteristic equation

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$$

Determine whether or not any of the roots of the charateristic equation lie outside the unit circle in the z plane. use the bilinear transformation and the Routh stability Criterion.

Sol: Let us substitute $\frac{w+1}{w-1}$ for z in the given characteristic equation

$$\left(\frac{w+1}{w-1}\right)^3 - 1.3\left(\frac{w+1}{w-1}\right)^2 - 0.08\left(\frac{w+1}{w-1}\right) + 0.24 = 0$$

Clearing the fractions by multiplying both sides of this last equation by $(w-1)^3$

$$-0.14w^3 + 1.06w^2 + 5.10w + 1.98 = 0$$

$$w^3 - 7.571w^2 - 36.43w - 14.14 = 0$$

Routh array

w^3	1	-36.43
w^2	-7.571	-14.14
w^1	-38.30	0
w^0	-14.14	

Since there is one sign change for the coefficients in the first column, there is one root in the right half of the w plane. hence the characteristic equation has one root outside the unit circle in the z plane. Therefore the system is unstable.