UNIT - II

Z-transformations

A mathematical tool commonly used for the analysis of discrete-time control system is the z-transform.

For continuous signals: → Laplace transform
&
→ continuous Fourier Transform.

For discrete signals: Discrete time Fourier transform for frequency analysis. and

z-transform: For discrete signal.

with the z-transform method, the solutions to linear differential equations become algebraic nature.

Syllabus: Z-transform - Theorems - Inverse z-transforms - Formulation of difference equations and solving - Block diagram representation - pulse transfer functions and finding open loop and closed loop responses.

Z-transform Definition:

The z-transform of time function $x(t)$ where t is nonnegative, or of a sequence of values $x(kT)$, where k takes zero or positive integers and T is the sampling period is defined by the following equation.

$$\boxed{Z[x(t)] = X(z) = Z[x(kT)] = \sum_{k=0}^{\infty} x(kT) z^{-k}} \rightarrow (1)$$

For a sequence of numbers $x(k)$, the z-transform is defined by,

$$\boxed{X(z) = Z[x(k)] = \sum_{k=0}^{\infty} x(k) z^{-k}} \rightarrow (2)$$

The equations (1) and (2) are called one-sided z-transform.

where $x(t)=0$ for $t<0$

$x(k)=0$ for $k<0$

$z=$ complex variable $=re^{j\theta}$

where $r=|z|$, and $\theta=\angle z$

Two-sided z-transform:

$$Z[x(t)] = X(z) = Z[x(kT)] = \sum_{k=-\infty}^{\infty} x(kT)\, z^{-k}$$

(or)

$$X(z) = Z[x(k)] = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}$$

The equation ① can be expanded as

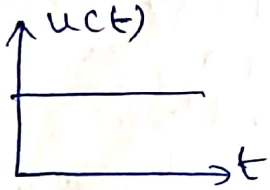
$$X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k}$$

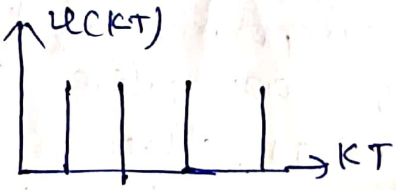
$$X(z) = x(0)z^{-0} + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots$$

The z^{-k} in this series indicates the position in time at which the amplitude $x(kT)$ occurs.

Z-transformation of unit-step function:

Note: $Z[x(t)] = X^*(s) = \sum_{k=0}^{\infty} x(kT)\, z^{-k}$





$Z[u(t)] = ZT[u(kT)] = ZT[(1)^k]$ (All are same)

$$= \sum_{k=0}^{\infty} x(kT)\cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} u(kT)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} (1)\, z^{-k}$$

$$= z^{0} + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

It is in the form of infinite series

$$= 1 + (z^{-1})^1 + (z^{-1})^2 + (z^{-1})^3 + \cdots$$

using Formula $\sum_{k=0}^{\infty} c^k = \frac{1}{1-c}$

$[\because 1 + c^1 + c^2 + c^3 + \cdots = \frac{1}{1-c}]$

so that

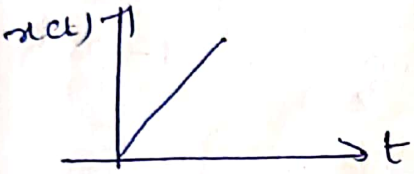
$X(z) = 1 + (z^{-1})^1 + (z^{-1})^2 + (z^{-1})^3 + \cdots$

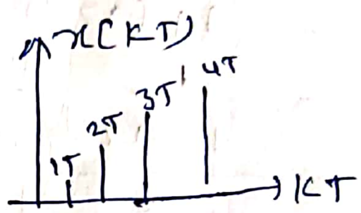
$$\boxed{X(z) = \frac{1}{1-z^{-1}}} = \frac{\text{First term}}{1 - \text{common Ratio}}$$

so

$$\boxed{Z[u(t)] = \frac{1}{1-z^{-1}} = \frac{1}{1-\frac{1}{z}} = \frac{1}{\frac{z-1}{z}} = \frac{z}{z-1}}$$

* For Ramp signal:





$Z[x(t)] = X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k}$

$\{x(kT) = kT$ for $kT \geq 0\}$

for $k = 0, 1, 2, 3, \ldots$

$Z[x(kT)] = X(z) = \sum_{k=0}^{\infty} kT z^{-k}$

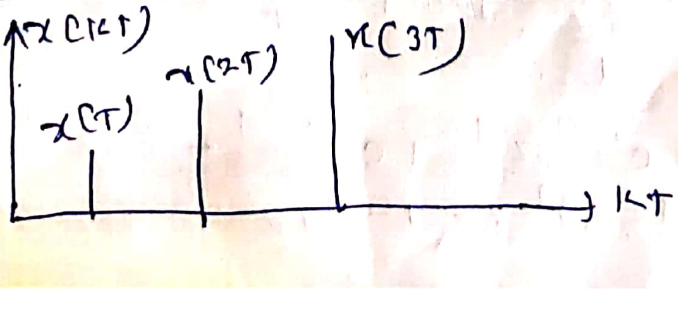
$= T \sum_{k=0}^{\infty} k z^{-k}$

$= T(z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots)$

$= T \frac{z^{-1}}{(1-z^{-1})^2} = \frac{T/z}{(1-\frac{1}{z})^2} = \frac{T/z}{(\frac{z-1}{z})^2}$

$$\boxed{X(z) = \frac{Tz^2/z}{(z-1)^2} = \frac{Tz}{(z-1)^2}} = Z[t]$$

$$\boxed{[\because x^{-1} + 2x^{-2} + 3x^{-3} + 4x^{-4} + \cdots = \frac{x^{-1}}{(1-x^{-1})^2}}$$



* Find z transform of a^k

$$x(k) = \begin{cases} a^k & k = 0,1,2,3\ldots \\ 0 & k<0 \end{cases}$$

$$X(z) = \sum_{k=0}^{\alpha} x(kT)\, z^{+k}.$$

$$X(z) = Z[a^k] = \sum_{k=0}^{\alpha} a^k z^{-k}$$

$$= a^0 z^{-0} + a^1 z^{-1} + a^2 z^{-2} + \ldots$$

$$= (az)^0 + ((az)^{-1})^1 + (az^{-1})^2 + (az^{-1})^3$$

$$= 1 + (az^{-1})^1 + (az^{-1})^2 + (az^{-1})^3 + \ldots$$

$$= \frac{1}{1-az^{-1}} = \frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}}$$

$$\boxed{X(z) = \frac{z}{z-a}}$$

* Find z transform of e^{-at}

$$x(t) = e^{-at}$$

$$x(kT) = e^{-akT}$$

$$Z[x(kT)] = Z[x(t)] = X(z) = \sum_{k=0}^{\alpha} x(kT) z^{-k}$$

$$X(z) = \sum_{k=0}^{\alpha} e^{-akT} z^{-k}$$

$$X(z) = e^{-0} z^{-0} + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + \ldots$$

$$X(z) = 1 + (e^{-aT} z^{-1})^1 + (e^{-aT} z^{-1})^2 + \ldots$$

$$\boxed{X(z) = \frac{1}{1-e^{-aT} z^{-1}} = \frac{1}{1-\frac{e^{-aT}}{z}} = \frac{z}{z-e^{-aT}}}$$

* Find z-transform of $e^{j\omega t}$

$$Z[e^{j\omega t}] = Z[e^{j\omega kT}] = \sum_{k=0}^{\alpha} x(kT) z^{-k}$$

$$= \sum_{k=0}^{\alpha} e^{j\omega kT} z^{-k}$$

$$= e^0 z^{-0} + e^{j\omega T} z^{-1} + e^{j\omega 2T} z^{-2} + e^{j3\omega T} z^{-3} + \ldots$$

$$= 1 + (e^{j\omega T} z^{-1})^1 + (e^{j\omega T} z^{-1})^2 + (e^{j\omega T} z^{-1})^3 + \ldots$$

$$= 1 + x^1 + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$= \frac{1}{1 - e^{j\omega T} z^{-1}} = \frac{1}{1 - \frac{e^{j\omega T}}{z}}$$

$$\boxed{= \frac{z}{z - e^{j\omega T}}}$$

$$\boxed{z[e^{j\omega t}] = \frac{z}{z - e^{j\omega T}}}$$

(or)

* $$z.T[e^{ak}] = \frac{z}{z - e^{aT}}$$

$$\boxed{z[e^{j\omega kT}] = \frac{z}{z - e^{j\omega T}}}$$

* z- transform of $\sin\omega t$

$x(t) = \sin\omega t$

$$x(t) = \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$z(x(t)) = z[\sin\omega t] = z(\sin\omega kT)$$

$$= z\left[\frac{e^{j\omega kT} - e^{-j\omega kT}}{2j}\right]$$

$$z(\sin\omega kT) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{j\omega kT} - e^{-j\omega kT}}{2j}\right) z^{-k}$$

$$= \frac{1}{2j}\left(\sum_{k=0}^{\infty} e^{j\omega kT} z^{-k}\right)$$

$$= \frac{1}{2j}\left[z(e^{j\omega kT}) - z(e^{-j\omega kT})\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right]$$

$$\left(\because z(a^k) = \frac{z}{z-a}\right)$$

$$= \frac{1}{2j}\left[\frac{z^2 - ze^{-j\omega T} - z^2 + ze^{j\omega T}}{z^2 - ze^{-j\omega T} - ze^{j\omega T} + e^{-j\omega T}e^{j\omega T}}\right]$$

$$= \frac{1}{2j}\left[\frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1}\right]$$

$$= \frac{1}{2j}\left[\frac{2z\left(\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right)}{z^2 - 2z\left(\frac{e^{-j\omega T} + e^{j\omega T}}{2}\right) + 1}\right]$$

$$= \frac{1}{2j}\left(\frac{2z \sin \omega T}{z^2 - 2z\cos\omega T + 1}\right)$$

$$= \frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$$

$$\boxed{Z[\sin \omega kT] = \frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}}$$

* obtain z transform of $x(t) = \cos \omega t$

$x(t) = \cos \omega t$

$$x(kT) = \cos \omega kT = \frac{e^{j\omega kT} + e^{-j\omega kT}}{2}$$

$$Z[x(t)] = Z[x(kT)] = Z[\cos \omega kT]$$

$$= Z\left[\frac{e^{j\omega kT} + e^{-j\omega kT}}{2}\right]$$

$$= \frac{1}{2} Z\left[e^{j\omega kT} + e^{-j\omega kT}\right]$$

$$\left[\because Z\{a^k\} = \frac{z}{z-a}\right]$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

$$= \frac{1}{2}\left(\frac{z^2 - ze^{-j\omega T} + z^2 - ze^{+j\omega T}}{z^2 - ze^{-j\omega T} - ze^{j\omega T} + e^{j\omega T}e^{-j\omega T}}\right)$$

$$= \frac{1}{2}\left(\frac{2z^2 - z(e^{-j\omega T} + e^{+j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} \right)$$

$$= \frac{1}{2}\left(\frac{2z^2 - 2z\frac{(e^{-j\omega T} + e^{+j\omega T})}{2}}{z^2 - z\frac{(e^{-j\omega T} + e^{j\omega T})}{2}2 + 1} \right)$$

$$= \frac{1}{2}\left(\frac{2z^2 - 2z\cos\omega T}{z^2 - z2\cos\omega T + 1} \right)$$

$$= \frac{1}{2}\left(\frac{2(z^2 - z\cos\omega T)}{z^2 - 2z\cos\omega T + 1} \right)$$

$$= \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}$$

$$\boxed{Z[\cos\omega t] = \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}}$$

properties and theorems:

Multiplication by a constant:

If $X(z)$ is the z transform of $x(t)$, then

$$Z[a x(t)] = a Z(x(t)) = a X(z)$$

a is constant

proof: $x(t) \xleftrightarrow{ZT} x[z]$

$$Z(a x(t)) = Z(a(x(kT)) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$Z(a x(t)) = \sum_{k=0}^{\infty} a x(kT) z^{-k}$$

$$= a \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= a Z(x(kT)) = a X(z)$$

$$\boxed{Z(a(x(t))) = a X(z)}$$

*2) linearity property:

$$x(KT) \xleftrightarrow{Z} X(z)$$

$$y(KT) \xleftrightarrow{Z} Y(z)$$

$$a\,x(KT) + b\,y(KT) \xleftrightarrow{Z} aX(z) + Y(z)b$$
$$aX(z) + b\,Y(z)$$

proof: $Z(x(KT)) = X(z) = \sum_{K=0}^{\infty} x(KT) z^{-K}$

$$Z[a x(KT) + b y(KT)] = \sum_{K=0}^{\infty} (a x(KT) z^{-K}) + (b y(KT) z^{-K})$$

$$= \sum_{K=0}^{\infty} a x(KT) z^{-K} + \sum_{K=0}^{\infty} b\, y(KT) z^{-K}.$$

$$\boxed{= aX(z) + bY(z)}$$

3) Multiplication by a^K:

$$x(KT) \xleftrightarrow{Z} X(z)$$

$$a^K x(KT) \xleftrightarrow{Z} X(a^{-1} z)$$

$$X(z) = \sum_{K=0}^{\infty} x(KT) z^{-K}$$

$$Z(a^K \cdot x(KT)) = \sum_{K=0}^{\infty} a^K x(KT) z^{-K}$$

$$= \sum_{K=0}^{\infty} x(KT) (a^{-1} z)^{-K}$$

$$= \sum_{K=0}^{\infty} x(KT) [a^{-1} z)^{-K}$$

$$\boxed{Z[a^K x(KT)] = X(a^{-1} z)}$$

4) Shifting theorem:

$$x(KT) \xleftrightarrow{Z} X(z)$$

$x(KT - nT)$ or $x(t - nT) \xleftrightarrow{Z} z^{-n} X(z)$

$x(KT + nT)$ or $x(t + nT) \xleftrightarrow{Z} z^{n}\left(X(z) - \sum_{K=0}^{n-1} x(KT) z^{-K}\right)$

↳ left shift

where n is a zero or positive integer.

proof: $Z[x(kT)] = \sum_{k=0}^{\infty} x(kT) z^{-k}$

$$Z[x(t-nT)] = \sum_{k=0}^{\infty} x(kT-nT) z^{-k}.$$

$$= \sum_{k=0}^{\infty} x(kT-nT) z^{-k}\cdot z^{-n}\cdot z^{n}$$

$$= z^{-n}\sum_{k=0}^{\infty} x(kT-nT) z^{-(k-n)}$$

Let $m = k-n$

$$= z^{-n}\sum_{k=0}^{\infty} x(mT) z^{-m}$$

$$= X(z) z^{-n} = z^{-n}X(z)$$

so that $\boxed{x(kT-nT) \xleftrightarrow{z} z^{-n}X(z)}$

Thus, Multiplication of a z-transform by z^{-n} has the effect of delaying the time function $x(t)$ by time nT. (That is, move the function to the right by time nT.).

proof for $Z[x(kT+nT)] = z^{+n}\left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}\right]$

$$Z[x(kT+nT)] = \sum_{k=0}^{\infty} x(kT+nT) z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT+nT) z^{-n} z^{n} z^{-k}$$

$$= z^{n}\sum_{k=0}^{\infty} x(kT+nT) z^{-(n+k)}$$

$$= z^{n}\left(\cancel{\sum_{k=0}^{\infty} x(kT+nT) z^{-(n+k)} + \sum_{k=0}^{n-1} x(kT) z^{-k} - \sum_{k=0}^{n-1} x(kT) z^{-k}}\right)$$

$$= z^{n}\left(\cancel{\sum_{k=0}^{\infty} x(kT+nT) z^{-(n+k)} + \sum_{k=0}^{n-1} x(kT) z^{-k} - \sum_{k=0}^{n-1} x(kT) z^{-k}}\right]$$

$$= z^{n}\left[\sum_{k=0}^{\infty} x(kT+nT) z^{-(n+k)} + \sum_{k=0}^{n-1} x(kT) z^{-k} - \sum_{k=0}^{n-1} x(kT) z^{-k}\right]$$

$$= z^n \left[\sum_{k=0}^{\infty} x(kT) z^{-k} - \sum_{k=0}^{n-1} x(kT) z^{-k} \right]$$

$$= z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k} \right)$$

so that

$$\boxed{Z[x(kT+nT)] = z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k} \right]}$$

(or)

$$Z[x(k+n)] = z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k} \right]$$

E.g: $Z[x(k+1)]$

Here $n=1$ hence

$$Z[x(k+1)] = z^1 \left(X(z) - \sum_{k=0}^{0} x(kT) z^{-k} \right]$$

$$= z\,[X(z) - x(0)]$$

$$= z\,X(z) - z\,x(0)$$

similary

$$Z[x(k+n)] = z^n X(z) - z^n x(0) - z^{n-1} x(1) - z^{n-2} x(2) \ldots - z\,x(n-1)$$

where n is a +ve integer.

Multiplication of the z transform $X(z)$ by z has the effect of advancing the signal $x(kT)$ by one step (1 sampling periods) and that Multiplication of the z transform $X(z)$ by z^{-1} has the effect of delaying the signal $x(kT)$ by one step (1 sampling period).

E.g: problem: Find z-transform $x(t) = 1(t-T)$

$$Z(x(t)) = Z(1(t-T))$$

$$Z(x(kT-nT)) = z^{-n} X(z)$$

$$Z[1(t-T)] = z^{-1} Z[1(t)]$$

$$= z^{-1} Z[1(kT)]$$

$$= z^{-1} \left[\sum_{k=0}^{\infty} z^{-k} x(kT) \right)$$

$$= z^{-1} \frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}}$$

* Find z transform of $z(1(t-4T))$

$$z(x(t-T)) = z^{-n} X(z)$$

$$z(1(t-4T)) = z^{-4} z[x(kT)]$$

$$= z^{-4} \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= z^{-4} \times \frac{1}{1-z^{-1}}$$

$$\boxed{z[1(t-4T)] = \frac{z^{-4}}{1-z^{-1}}}$$

* obtain the z transform of a^{k-1}

$$f(a) = \begin{cases} a^{k-1} & k = 1,2,3,\ldots \\ 0 & k \le 0 \end{cases}$$

$z[a^{k-1}]$ using time shifting property

$$\boxed{z[x(t-T)] = z^{-n} X(z)}$$

$$z[a^{k-1}] = z^{-n} z[a^k]$$

$$= z^{-n} \cdot \frac{1}{1-az^{-1}}$$

$$\boxed{z(a^{k-1}) = z^{-1} \times \frac{1}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}}$$

Complex Translation theorem: If $x(t)$ has the z transform $X(z)$, then the z transform of $e^{-at}x(t)$ can be given by $X(ze^{aT})$ this is known as the complex translation theorem.

$$x(t) \overset{z}{\longleftrightarrow} X(z)$$

$$e^{-at}x(t) \overset{z}{\longleftrightarrow} X(ze^{aT})$$

prove this: $z[x(t)] = X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$

$$z[e^{-at}x(t)] = \sum_{k=0}^{\infty} e^{-akT} x(kT) z^{-k}$$

$$Z[e^{-at}x(t)] = \sum_{k=0}^{\infty} x(kT)\, e^{-akT} z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT)\, (z e^{aT})^{-k}$$

$$\boxed{Z(e^{-at}x(t)) = X(z e^{aT})}$$

so that replacing z in $X(z)$ by ze^{aT} gives z transfor of $e^{-at}x(t)$

Eg) obtain z transform of $e^{-at}\sin\omega t$ and $e^{-at}\cos\omega t$

using complex translation theorem

$$Z(e^{-at} \cancel{\cos\omega t}\, x(t)) = X(ze^{aT})$$

$$Z[e^{-at}x(t)] = X(ze^{aT})$$

$*$ $Z[e^{-at}\sin\omega t]$:

$$Z(\sin\omega t) = \frac{z^{-1}\sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

Now for

$$Z(e^{-at}\sin\omega t) = \frac{(ze^{aT})^{-1}\sin\omega T}{1 - 2(ze^{aT})^{-1}\cos\omega T + (ze^{aT})^{-2}}$$

$$\boxed{Z(e^{-at}\sin\omega t) = \frac{z^{-1}e^{-aT}\sin\omega T}{1 - 2z^{-1}e^{-aT}\cos\omega T + e^{-2aT}z^{-2}}}$$

For $Z(e^{-at}\cos\omega t)$

$$Z(\cos\omega t) = \frac{1 - z^{-1}\cos\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

$$Z(e^{-at}\cos\omega t) = \frac{1 - (ze^{aT})^{-1}\cos\omega T}{1 - 2(ze^{aT})^{-1}\cos\omega T + (ze^{aT})^{-2}}$$

$$\boxed{= \frac{1 - z^{-1}e^{-aT}\cos\omega T}{1 - 2z^{-1}e^{-aT}\cos\omega T + z^{-2}e^{-2aT}}}$$

* obtain the z transform of te^{-at} using complex translation theorem

$$z[e^{-at}x(t)] = X[ze^{aT}]$$

$$z(t) = \frac{Tz}{(z-1)^2}$$

$$\boxed{z(te^{-at}) = \frac{T(ze^{aT})}{(ze^{aT}-1)^2}}$$

complex differentiation theorem:

$$x(kT) \overset{z}{\longleftrightarrow} X(z)$$

$$z(kx(k)) \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z)$$

proof:

$$\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} x(k)\frac{d}{dz}(z^{-k})$$

$$= \sum_{k=0}^{\infty} x(k)(z^{-k})(-k)$$

$$= \sum_{k=0}^{\infty} x(k)(-k)z^{-k-1}$$

$$\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} x(k)(-k)z^{-k}\cdot z^{-1}$$

$$\frac{d}{dz}X(z) = \frac{-1}{z}\sum_{k=0}^{\infty} x(k)(k)z^{-k}$$

$$-z\frac{d}{dz}X(z) = \sum_{k=0}^{\infty} k\,x(k)z^{-k}$$

$$-z\frac{d}{dz}X(z) = z[kx(k)]$$

$$\boxed{\text{so that } z(kx(k)) = -z\frac{d}{dz}X(z)}$$

$$\boxed{\text{similary } z[k^m x(k)] = (-z)^m\frac{d^m}{dz^m}X[z]}$$

problem: Find Z.T of t^2e^{-at} using complex translation theorem.

$$z[e^{-at}x(t)] = X(ze^{-aT})$$

Now $z(t^2e^{-at}) = z(t\,te^{-at})$

$z(te^{-at}) \Rightarrow$ by complex translation theorem.

$Z(te^{-at})$

$Z(t) = \frac{Tz}{(z-1)^2}$

$$\boxed{Z(te^{-at}) = \frac{T(ze^{+aT})}{(ze^{+aT}-1)^2}}$$

Now

$\Rightarrow Z(t\,te^{-at})$

using complex differention theorem.

$Z(k\,x(k)) = -z\frac{d}{dz}X(z)$

$Z(t\,te^{-at}) = -z\frac{d}{dz}\left(\frac{Tze^{aT}}{(ze^{aT}-1)^2}\right)$

$= -zT\frac{d}{dz}\left(\frac{ze^{aT}}{(ze^{aT}-1)^2}\right)$

$$\boxed{Z(t\,te^{-at}) = \frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}}$$

The above problem can also be solved by:

$Z(e^{-at}) = Z(e^{-akT}) = \frac{z}{z-e^{-aT}}$

Now

$Z(t^2e^{-at}) = Z((kT)^2e^{-akT}) =$

$= Z(k^2T^2e^{-akT})$

$= T^2 Z(k^2e^{-akT})$

$= T^2(-z)^2\frac{d^2}{(dz)^2}\left(\frac{z}{z-e^{-aT}}\right)$

$= (T^2)(-z)^2\left(\left(\frac{d}{dz}\right)^2\left(\frac{z}{z-e^{-aT}}\right)\right)$

$$\boxed{Z(t^2e^{-at}) = \frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}}$$

Initial value theorem:

If $x(t)$ has the z-transform $X(z)$ and if $\lim_{z\to\infty} X(z)$ exists, then the initial value $x(0)$ of $x(t)$ or $x(kT)$ is given by.

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

proof: $X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k}$

$$X(z) = x(0)\, z^{-0} + x(T) z^{-1} + x(2T) z^{-2} + x(3T) z^{-3} + \cdots$$

$$X(z) = x(0) + \frac{x(T)}{z} + \frac{x(2T)}{z^2} + \frac{x(3T)}{z^3} + \cdots$$

Apply limits

$$\lim_{z\to\infty} (X(z)) = \lim_{z\to\infty} \left[x(0) + \frac{x(T)}{z} + \frac{x(2T)}{z^2} + \cdots \right]$$

$$\lim_{z\to\infty} X(z) = x(0) + \frac{x(T)}{\infty} + \frac{x(2T)}{\infty^2} + \cdots$$

$$\boxed{\lim_{z\to\infty} X(z) = x(0) + 0 + 0 + 0 \cdots}$$

so that $\boxed{\lim_{z\to\infty} X(z) = x(0)}$

* Determine the initial value $x(0)$ if the z-transform of $x(t)$ given by.

$$X(z) = \frac{(1-e^{-T})\, z^{-1}}{(1-z^{-1})\,(1-e^{-T} z^{-1})}$$

$$x(0) = \lim_{z\to\infty} \frac{(1-e^{-T})\, z^{-1}}{(1-z^{-1})\,(1-e^{-T} z^{-1})}$$

$$= \frac{(1-e^{-T})\left(\frac{1}{\infty}\right)}{\left(1-\frac{1}{\infty}\right)\left(1-e^{-T}/\infty\right)} = 0$$

Scanned by CamScanner

Final value theorem:

If $x(kT)$ has the z transform $X(z)$, then the final value of $x(kT)$, that is, the value of $x(kT)$ as k approaches infinity, can be given by:

$$x(kT) \overset{z}{\longleftrightarrow} X(z)$$

$$\boxed{\lim_{k\to\infty} x(kT) = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]}$$

$$x(\infty) = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]$$

proof:

we know that

$$Z[x(kT)] = X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} \rightarrow ①$$

$$Z[x(kT-1)] = z^{-1} X(z) = \sum_{k=0}^{\infty} x(kT-1) z^{-k} \rightarrow ②$$

① − ②:

$$\sum_{k=0}^{\infty} x(kT) z^{-k} - \sum_{k=0}^{\infty} x(kT-1) z^{-k} = X(z) - z^{-1}X(z)$$

$$\sum_{k=0}^{\infty} \left[x(kT) z^{-k} - x(kT-1) z^{-k}\right] = X(z)\left[1 - z^{-1}\right]$$

Apply limits $\lim_{z\to 1}$

$$\lim_{z\to 1}\left[\sum_{k=0}^{\infty} x(kT) z^{-k} - x(kT-1) z^{-k}\right] = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]$$

$$\sum_{k=0}^{\infty} x(kT)\, 1^{-k} - x(kT-1)\, 1^{-k}\Big] = \lim_{z\to 1}\left[(1-z^{-1}) X(z)\right]$$

$$\sum_{k=0}^{\infty} x(kT)\, 1^{-k} - x(kT-1)\, 1^{-k}\Big] = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]$$

$$\left[x(0) - x(-1) + x(1T) - x(1T-1) + x(2T) - x(2T-1) + \cdots\right] = \lim_{z\to 1}\left[(1-z^{-1}) X(z)\right]$$

$$\left[x(0) - x(-1) + x(1) - x(0) + x(2) - x(1) + \cdots\right] = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]$$

$$x(\alpha) = \lim_{k\to\alpha} x(kT)$$

(or)

$$\boxed{x(\alpha) = \lim_{k\to\alpha} x(k)}$$

so that $\lim_{k\to\alpha} (x(k)) = \lim_{z\to 1} [(1-z^{-1})x(z)]$

$$\boxed{\lim_{k\to\alpha} [x(k)] = \lim_{z\to 1} [(1-z^{-1})x(z)]}$$

Determinal Final value of $a>0$

$$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$$

using Final value theorem.

$$x(\alpha) = \lim_{k\to\alpha} x(k) = \lim_{z\to 1} [(1-z^{-1})\, x(k)]$$

$$x(\alpha) = \lim_{z\to 1} [(1-z^{-1})] \left(\left(\frac{1}{1-z^{-1}}\right) - \frac{1}{(1-e^{-aT}z^{-1})} \right)$$

$$\cancel{x(\alpha)} = \left[\cancel{\left(0+\frac{1}{1}\right)} \right] \left[\left(\cancel{\frac{1}{1-1}} \right) - \frac{1}{\left(\cancel{1-e^{-aT}}\right)} \right]$$

$$x(\alpha) = \lim_{z\to 1} [(1-z^{-1})] \left[\frac{(1-e^{-aT})z^{-1} - (1-z^{-1})}{(1-z^{-1})\,(1-e^{-aT}z^{-1})} \right]$$

$$x(\alpha) = \lim_{z\to 1} \left[\frac{(1-e^{-aT})z^{-1} - (1-z^{-1})}{(1-e^{-aT}z^{-1})} \right]$$

$$x(\alpha) = \frac{(1-e^{-aT})(1) - (0)}{(1-e^{-aT})}$$

$$\boxed{x(\alpha) = 1}$$

<u>Inverse z-transform</u>: When $X(z)$, the z transform of $x(kT)$ or $x(k)$, is given the operation that determines the corresponding $x(kT)$ or $x(k)$ is called the inverse z-transform.

The notation for inverse z-transform is z^{-1}. i,e $\boxed{z^{-1}[X(z)] = x(kT) \text{ or } x(k)}$

Methods:

1) Direct division Method.
2) computation Method.
3) partial-fraction expansion Method.
4) Inverse Integral Method.

poles and zeros in z-plane:

$X(z)$ may be written as.

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \dots + b_M}{z^n + a_1 z^{n-1} + \dots + a_n}$$

(or) $M \le n$

$$X(z) = \frac{b_0 (z - z_1)(z - z_2) \dots (z - z_M)}{(z - P_1)(z - P_2) \dots (z - P_n)}$$

where $z_1, z_2, z_3 \dots z_m$ are the zeros

$P_1, P_2, P_3 \dots P_n$ are the poles.

In control engineering and signal processing $X(z)$ is a frequently expressed as a ratio of polynomials in z^{-1} as follows.

$$X(z) = \frac{b_0 z^{-(n-M)} + b_1 z^{-(n-M+1)} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_0 z^{-n}}$$

where z^{-1} is the unit delay operator.

* Find poles and zeros:

$$X(z) = \frac{z^2 + 0.5z}{z^2 + 3z + 2}$$

sol:

$$X(z) = \frac{z(0.5 + z)}{(z+1)(z+2)} \rightarrow \text{Ratio of polynomials in } z.$$

now: zeros: $z(0.5 + z) = 0$

$\therefore z_1 = 0$ and $z_2 = -0.5$

poles: $(z+1)(z+2) = 0$

$\cancel{P_1 = -1}$ $z + 1 = 0$

$\boxed{z = -1}$ pole

$(z + 2) = 0$

$\boxed{z = -2}$ pole

Direct Division Method: In this method we obtain the inverse z-transform by expanding $X(z)$ into an infinite power series in z^{-1}.

$$X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k}$$

$$= x(0) + x(T)z^{-1} + x(2T)z^{-2} + x(3T)z^{-3} + \cdots + x(kT)z^{-k}$$

The $x(kT)$ is the coefficient of the z^{-k} term. Hence the values of $x(kT)$ or $x(k)$ for $k = 0, 1, 2, 3 \ldots$ can be determined by inspection.

Find inverse z-transform of $x(k)$ for $k = 0, 1, 2, 3, 4$ when $X(z)$ is given by.

$$X(z) = \frac{10z+5}{(z-1)(z-0.2)}$$

sol: Rewrite $X(z)$ as Ratio of polynomials in z^{-1}

$$X(z) = \frac{10z+5}{z^2-0.2z-z+0.2}$$

$$= \frac{10z+8}{z^2-1.2z+0.2}$$

$$= \frac{10z+5/z^2}{z^2-1.2z+0.2/z^2}$$

$$= \frac{10z^{-1}+5z^{-2}}{1-1.2z^{-1}+0.2z^{-2}}$$

$$
\begin{array}{l}
1-1.2z^{-1}+0.2z^{-2}\,\big)\; 10z^{-1}+5z^{-2}\;\big(10z^{-1}+17z^{-2}+18.4z^{-3}+18.68z^{-4} \\
\quad\quad\quad\quad\quad\quad\quad\; \underline{\overset{(-)}{}10z^{-1} \overset{(+)}{-} 12z^{-2} \overset{(-)}{+} 2z^{-3}} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 17z^{-2}-2z^{-3} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{17z^{-2} \overset{(+)}{-} 20.4z^{-3} \overset{(-)}{+} 3.4z^{-4}} \\
\quad\quad\quad\quad 18.4z^{-3}-3.4z^{-4} \\
\quad\quad\quad\quad \underline{\overset{(-)}{}18.4z^{-3} \overset{(+)}{-} 22.08z^{-4} \overset{(-)}{+} 3.68z^{-5}} \\
\quad\quad\quad\quad\quad\quad\quad\quad 18.68z^{-4}-3.68z^{-5} \\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{18.68z^{-4}-22.416z^{-5}+3.736z^{-6}}
\end{array}
$$

Thus $X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \ldots$

$= x(0)z^{-1} + x(1T)z^{-2} + x(3T)z^{-3} + \ldots$

$x(0) = 0$

$x(1) = 1$

$x(2) = 17$

$x(3) = 18.4$

$x(4) = 18.68$

$x(kT) = \{0, 1, 17, 18.4, 18.68 \ldots\}$

from the above Example it is clear that ~~the~~ if only the first several terms of the sequence are desired.

*2) Find inverse z-transfor of

$$X(z) = \frac{2+3z^{-1}}{(1+z^{-1})\left(1+0.25z^{-1}-\frac{z^{-2}}{8}\right)}$$

$$X(z) = \frac{2+z^{-1}}{(1+z^{-1})(1+0.25z^{-1}-0.125z^{-2})}$$

$$X(z) = \frac{2+z^{-1}}{1+z^{-1}+0.25z^{-1}+0.25z^{-2}-0.125z^{-2}-0.125z^{-3}}$$

$$= \frac{2+3z^{-1}}{1+1.25z^{-1}+0.125z^{-2}-0.125z^{-3}}$$

$$1+1.25z^{-1}+0.125z^{-2}-0.125z^{-3}\,)\; 2+3z^{-1}\; (2+0.5z^{-1}$$

$$2+2.5z^{-1}+0.25z^{-2} - 0.25z^{-3}$$

$$(-)\;(-)\;(+)$$

$$+0.5z^{-1}-0.25z^{-2}+0.25z^{-3}$$

$$0.5z^{-1}+0.625z^{-2}+0.0625z^{-3}-0.0625z^{-4}$$

$$(-)\;(-)\;(-)\;(+)$$

$$-0.875z^{-2}+0.1875z^{-3}+0.0625z^{-4}$$

$X(z) = 2 + 0.5z^{-1} + \ldots$

$X(z) = 2+0.5z^{-1} - 0.875z^{-2} + \ldots$

$x(0) = 2$

$x(1) = 0.5$

$x(2) = -0.875$

Scanned by CamScanner

Note: useful formulas:

$$\frac{Az}{z-1} \xleftrightarrow{z^{-1}} A(1)^k$$

$$\frac{Bz}{(z-1)^2} \xleftrightarrow{z^{-1}} \frac{B}{T}t$$

$$\frac{Cz}{z-a} \xleftrightarrow{z^{-1}} C.a^k$$

$$\frac{Dz}{z^2-cz+d} \longleftrightarrow De^{-at}\sin\omega_1 t$$

complex conjugate

Find inverse z-transform of $X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

$$\frac{X(z)}{z} = \frac{1-e^{-aT}}{(z-1)(z-e^{-aT})}$$

$$Z^{-1}(X(z)) = x(kT) = Z^{-1}\left(\frac{1-e^{-aT}}{(z-1)(z-e^{-aT})}\right)$$

using partial fractions.

$$\frac{X(z)}{z} = \frac{A}{z-1} + \frac{B}{z-e^{-aT}} \longrightarrow (1)$$

$$\frac{1-e^{-aT}}{(z-1)(z-e^{-aT})} = \frac{A}{z-1} + \frac{B}{z-e^{-aT}}$$

$$A = \left.\frac{1-e^{-aT}}{z-e^{-aT}}\right|_{z=1} = \frac{1-e^{-aT}}{1-e^{-aT}} = 1$$

$$\boxed{A=1}$$

$$B = \left.\frac{1-e^{-aT}}{(z-1)}\right|_{z=e^{-aT}}$$

$$B = \frac{1-e^{-aT}}{e^{-aT}-1} = \frac{1-e^{-aT}}{-(1-e^{-aT})} = -1$$

From (1):

$$\frac{X(z)}{z} = \frac{1}{z-1} + \frac{-1}{z-e^{-aT}}$$

$$X(z) = \frac{z}{z-1} + \frac{-z}{z-e^{-aT}}$$

$$Z^{-1}[X(z)] = x(kT) = Z^{-1}\left[\frac{z}{z-1}\right] + -Z^{-1}\left[\frac{z}{z-e^{-aT}}\right]$$

$$\boxed{x(kT) = 1 - e^{-akT}} \quad , k = 0, 1, 2, 3 \ldots$$

$$\left[\because \; Z[1] = \frac{z}{z-1}\right.$$

$$z^{-1}\left[\frac{z}{z-1}\right] = 1$$

$$\left[Z[a^k] = \frac{z}{z-a} \right]$$

$$\left. Z[e^{-akT}] = \frac{z}{z-e^{-aT}} \right]$$

* $$F(z) = \frac{z-4}{(z-1)(z-2)^2}$$

$$\frac{z-4}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{(z-2)^2} + \frac{C}{z-2} \longrightarrow (1)$$

$$A = \left.\frac{z-4}{(z-2)^2}\right|_{z=1} = \frac{1-4}{(1-2)^2} = \frac{-3}{(-1)^2}$$

$$\boxed{A = -3}$$

$$B = \left.\frac{z-4}{z-1}\right|_{z=2} = \frac{2-4}{2-1} = \frac{-2}{1}$$

$$\boxed{B = -2}$$

$$c = \frac{d}{dz}\left[\frac{z-4}{z-1}\right]_{z=2} = \frac{(z-1)(1)-(z-4)(1)}{(z-1)^2}$$

$$c = \left.\frac{z-1-z+4}{(z-1)^2}\right|_{z=2}$$

$$\boxed{c = \frac{2-1-2+4}{(2-1)^2} = \frac{3}{1} = 3}$$

<u>FROM (1)</u>:

$$\frac{z-4}{(z-1)(z-2)^2} = \frac{-3}{z-1} + \frac{-2}{(z-2)^2} + \frac{3}{z-2}$$

$$x(kT) = z^{-1}\left[\frac{-3}{z-1}\right] + z^{-1}\left[\frac{-2}{(z-2)^2}\right] + z^{-1}\left[\frac{3}{z-2}\right]$$

$$x(kT) = -3z^{-1}\left[\frac{z \cdot z^{-1}}{z-1}\right] + z^{-1}\left[\frac{-2z z^{-1}}{(z-2)^2}\right] + z^{-1}\left[\frac{3z \cdot z^{-1}}{z-2}\right]$$

$$x(kT) = -3\,[z^{-1}u(k)] - \mathcal{Z}^{-1}\left[\frac{2z\,z^{-1}}{(z-2)^2}\right] + z^{-1}\left[\frac{3z\cdot z^{-1}}{z-2}\right]$$

$$x(kT) = -3u(k-1) - [z^{-1}(k2^k)] + 3(2^k)z^{-1}$$

$$\boxed{x(kT) = -3u(k-1) - (k-1)2^{k-1} + 3(2^{k-1})}$$

$$(\because \mathcal{Z}(z^{-1}X(z)) = Z[x(kT-nT)])$$

$$\therefore \left[\mathcal{Z}\left(\frac{az}{(z-a)^2}\right) = Z(ka^k)\right]$$

<u>HW</u>: $X(z) = \dfrac{z}{(z-1)(z-2)}$

$$\boxed{x(kT) = 2^k - (1)^k}$$

* 3) $X(z) = \dfrac{10}{(z-1)(z-2)}$

$$x(kT) = z^{-1}(x(z)) = z^{-1}\left[\frac{10}{(z-1)(z-2)}\right]$$

$$\frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$A = \left.\frac{10}{z-2}\right|_{z=1} = \frac{10}{1-2} = -10$$

$$B = \left.\frac{10}{z-1}\right|_{z=2} = \frac{10}{2-1} = 10$$

$$x(kT) = \mathcal{Z}^{-1}\left[\frac{-10}{z-1}\right] + z^{-1}\left[\frac{10}{z-2}\right]$$

$$= \mathcal{Z}^{-1}\left[\frac{-10z\,z^{-1}}{z-1}\right] + z^{-1}\left[\frac{10z\,z^{-1}}{z-2}\right]$$

$$= \cancel{\mathcal{Z}^{-1}} \quad \begin{cases} x(kT) \longleftrightarrow X(z) \\ x(kT-nT) \longleftrightarrow z^{-n}X(z) \end{cases}$$

$$\boxed{x(kT) = -10\,u(kT-1) + 2^{k-1}}$$ Answer.

<u>problem</u>: $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

$$\frac{X(z)}{z} = \frac{z}{z^2 - z + 0.5}$$

$$\frac{z}{z^2 - z + 0.5} = \frac{z}{(z-0.5-j0.5)(z-0.5+j0.5)}$$

$$= \frac{A}{z-0.5-j0.5} + \frac{A^*}{z-0.5+j0.5}$$

$$A = \left.\frac{z}{z-0.5+j0.5}\right|_{z=0.5+j0.5}$$

$$A = \frac{0.5+j0.5}{0.5+j0.5-0.5+j0.5}$$

$$A = 0.5-j0.5$$

$$A^* = (0.5-j0.5)^* = 0.5+j0.5$$

$$\frac{X(z)}{z} = \frac{0.5-j0.5}{z-0.5-j0.5} + \frac{0.5+j0.5}{z-0.5+j0.5}$$

$$X(z) = \frac{(0.5-j0.5)z}{z-j0.5-0.5} + \frac{z(0.5+j0.5)}{z-0.5+j0.5}$$

$$\left[\because\ x(z) = z^{-1}\left[\frac{z}{z-a}\right] = a^k \right.$$
$$\left. \cancel{z}[a^k]z \right]$$

$$z^{-1}[X(z)] = Z^{-1}\left[\frac{(0.5-j0.5)z}{z-(0.5+j0.5)}\right] + \frac{(0.5+j0.5)z}{z[-(0.5-j0.5)]}$$

$$\boxed{x(kT) = (0.5-j0.5)(0.5+j0.5)^k + (0.5+j0.5)(0.5-j0.5)^k}$$

Inversion Integral Method:

The inversion integral for the z transform $X(z)$ is given by.

$$z^{-1}[X(z)] = x(kT) = x(k) = \frac{1}{2\pi j}\oint_c X(z)z^{k-1}dz$$

where; c = circle with center at origin of the z-plane such that all poles of $X(z)z^{k-1}$ are inside it.

equation for giving the inverse z-transform interms of residues:

$$x(kT) = x(k) = k_1 + k_2 + \cdots + k_m$$
$$= \sum_{i=1}^{m} [\text{Residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1}]$$

Where $k_1, k_2 \ldots k_m$ denote the residues of $X(z)z^{-k}$ at poles at $z_1, z_2 \ldots z_m$.

If $X(z) z^{-k}$ contains a simple pole $z = z_i$ then the corresponding residue K is given by.

$$K = \lim_{z \to z_i} \left[(z - z_i) X(z) z^{k-1} \right]$$

If $X(z) z^{k-1}$ contains a multiple pole z_j of order q, then the residue K is given by.

$$\boxed{K = \frac{1}{(q-1)!} \lim_{z \to z_j} \frac{d^{q-1}}{dz^{q-1}} \left((z - z_j)^q X(z) z^{k-1} \right)}$$

problem: obtain $x(kT)$ by using inversion integral method when $X(z)$ is given by.

$$X(z) = \frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})} \rightarrow ①$$

$$x(kT) = \frac{1}{2\pi j} \oint_c X(z) z^{k-1} \, dz$$

equation ① can be written as.

$$z^{k-1} X(z) = \frac{z(1 - e^{-aT}) z^{k-1}}{(z-1)(z - e^{-aT})}$$

$$X(z) z^{k-1} = \frac{(1 - e^{-aT}) z^k}{(z-1)(z - e^{-aT})}$$

poles are: $(z-1)(z - e^{-aT}) = 0$

$z = 1$ and $z = e^{-aT}$

Two are simple poles, hence it consists two residues K_1 and K_2.

$$K_1 = \lim_{z \to z_i} \left[(z - z_i) X(z) z^{k-1} \right]$$

$$K_1 = \lim_{z \to 1} \left[(z-1) \frac{(1 - e^{-aT}) z^k}{(z-1)(z - e^{-aT})} \right]$$

$$K_1 = \lim_{z \to 1} \left[\frac{(1 - e^{-aT}) z^k}{(z - e^{-aT})} \right] = \frac{(1 - e^{-aT}) z^k}{(1 - e^{-aT})}$$

$$\boxed{K_1 = 1^k = 1}$$

$$K_2 = \lim_{z \to e^{-aT}} \left[(z - e^{-aT}) \frac{(1 - e^{-aT}) z^k}{(z-1)(z - e^{-aT})} \right]$$

$$K_2 = \frac{(1 - e^{-aT}) e^{-aTk}}{(e^{-aT} - 1)} = -e^{-akT}$$

Hence $x(kT) = K_1 + K_2 = 1 - e^{-akT}$, $k = 0, 1, 2, \ldots$

* obtain inverse z tranform of

$$X(z) = \frac{z^2}{(z-1)^2 \cdot (z-e^{-aT})}$$

$$X(z) z^{k-1} = \frac{z^2 z^{k-1}}{(z-1)^2 (z-e^{-aT})}$$

$$X(z) z^{k-1} = \frac{z^{k+1}}{(z-1)^2 (z-e^{-aT})}$$

poles are: $(z-e^{-aT}) \cdot (z-1)^2 = 0$

$z_i = e^{-aT}$ and $z_j = 1$

simple pole Multiple pole

$$K_1 = \lim_{z \to z_i} [(z-z_i) X(z) z^{k-1}]$$

$$K_1 = \lim_{z \to e^{-aT}} \left[(z-e^{-aT}) \frac{z^{k+1}}{(z-1)^2 (z-e^{-aT})} \right]$$

$$\boxed{K_1 = \frac{e^{-aT(k+1)}}{(e^{-aT}-1)^2}}$$

$$K_2 = \lim_{z \to z_j} \left[\quad K_2 = \frac{1}{(q-1)!} \lim_{z \to z_j} \frac{d^{q-1}}{dz^{q-1}} \left[(z-z_j)^q \times X(z) z^{k-1} \right] \right.$$

$$K_2 = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 \frac{z^{k+1}}{(z-1)^2 (z-e^{-aT})} \right]$$

$$K_2 = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[\frac{z^{k+1}}{(z-e^{-aT})} \right]$$

$$K_2 = \lim_{z \to 1} \left[\frac{(z-e^{-aT})(k+1) z^{k+1-1} - (z^{k+1})(1)}{(z-e^{-aT})^2} \right]$$

$$K_2 = \lim_{z \to 1} \left[\frac{(z-e^{-aT})(k+1) z^k - (z^{k+1})}{(z-e^{-aT})^2} \right]$$

$$K_2 = \frac{(1-e^{-aT})(k+1)(1)^k - 1^{k+1}}{(1-e^{-aT})^2}$$

$$K_2 = \frac{(k+1)(1-e^{-aT}) - 1}{(1-e^{-aT})^2}$$

Hence $x(kT) = k_1 + k_2$

$$x(kT) = \frac{e^{-aT(k+1)}}{(e^{-aT}-1)^2} + \frac{(k+1)(1-e^{-aT})-1}{(1-e^{-aT})^2}$$

Differential equations solving:

Discrete Function	Z-Transform
$x(k+4)$	$z^4X(z) - z^4x(0) - z^3x(1) - z^2x(2) - zx(3)$
$x(k+3)$	$z^3X(z) - z^3x(0) - z^2x(1) - zx(2)$
$x(k+2)$	$z^2X(z) - z^2x(0) - z^1x(1)$ $[\because x(kT+nT) = z^n[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}]]$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1}X(z)$
$x(k-2)$	$z^{-2}X(z)$
$x(k-3)$	$z^{-3}X(z)$
$x(k-4)$	$z^{-4}X(z)$

$$\therefore x(kT+nT) = z^n[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}]$$

$$x(k+4) = z^4(X(z) - \sum_{k=0}^{4-1} x(kT)z^{-k})$$

$$= z^4(X(z) - [x(0) - x(1)z^{-1} - x(2)z^{-2} - x(3)z^{-3}]$$

$$= z^4X(z) - z^4x(0) - z^3x(1) - z^2x(2) - zx(3)$$

problem: solve the following differential equation by use of the z-transform method.

$x(k+2)+3x(k+1)+2x(k)=0, \quad x(0)0, \quad x(1)=1$

First from left shift theorem.

$$z[x(kT+nT)] = z^n\left[x(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$$

$$z[x(k+2)] = z^2\left[x(z) - \sum_{k=0}^{2-1=1} x(kT)z^{-k}\right]$$

$$z[x(k+2)] = z^2[x(z) - x(0)z^{-0} - x(1)z^{-1})$$

$$z[x(k+2)] = z^2x(z) - z^2x(0) - z^1x(1)$$

$$z[x(k+1)] = zx(z) - zx(0)$$

so that:

Apply z-transform

$$z[x(k+2)+3x(k+1)+2x(k)] = 0$$

$$z^2x(z) - z^2x(0) - zx(1) + 3z^1x(z) - 3zx(0)$$
$$+2x(z) = 0$$

$$z^2x(z) - z^2(0) - z(1) + 3zx(z) - 3zx(0)$$
$$+2x(z) = 0$$

$$x(z)(z^2+3z+2) = z$$

$$\boxed{x(z) = \frac{z}{z^2+3z+2}}$$

Now Apply inverse z-transform.

$$\frac{x(z)}{z} = \frac{1}{z^2+3z+2}$$

using partial fractions.

$$\frac{x(z)}{z} = \frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \quad \to (1)$$

$$A = \left.\frac{1}{z+2}\right|_{z=-1} = \frac{1}{-1+2} = 1$$

$$B = \left.\frac{1}{z+1}\right|_{z=-2} = \frac{1}{-2+1} = -1$$

From (1):

$$\frac{x(z)}{z} = \frac{1}{z+1} - \frac{1}{z+2}$$

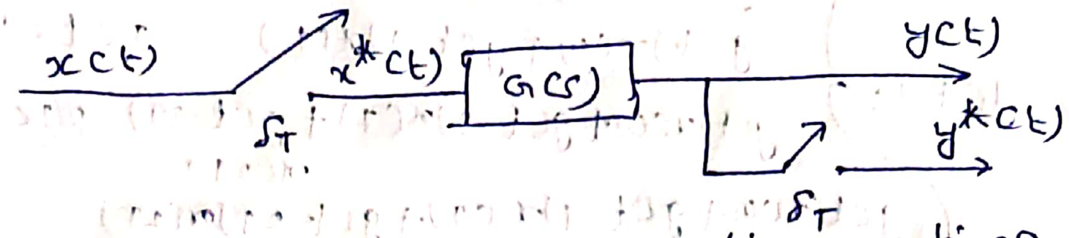
$$x(z) = \frac{z}{z+1} + \frac{-z}{z+2}$$

$$z^{-1}[x(z)] = z^{-1}\left[\frac{z}{z+1}\right] - z^{-1}\left[\frac{z}{z+2}\right]$$

$$x(kT) = (-1)^k - (-2)^k \quad , k=0,1,2,\ldots$$

pulse transfer function:

convolution summation:



consider the response of a continuous-time system driven by an Impulse-sampled signal. Assume that $x(t) = 0$ for $t \leq 0$

when the impulse sampled signal $x^*(t)$ is applied to the continuous time system it produces continuous-time signal $y(t)$.

If at the output there is another sampler which is synchronized in phase with the input sampler and operates at the same sampling period

Assume $y(t) = 0$ for $t < 0$

The z-tranform of $y(t)$ is

$$Z[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k} \rightarrow (1)$$

For the continuous time-system the output $y(t)$ of a system is related to the Input $x(t)$ is generally expressed by using convolution integral.

$$\text{The output } y(t) = \int_0^t g(t-\tau)x(\tau)\,d\tau$$

$$= \int_0^t x(t-\tau) g(\tau)\,d\tau \rightarrow (2)$$

where $g(\tau)$ = weighting function or Impulse function of the system.

similarly, the convolution summation for discrete-time system is given as, and which is similar to convolution integral

$$x^*(t) = \sum_{k=0}^{\infty} x(t)\delta(t-kT) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) \rightarrow (2)$$

The response $y(t)$ of the system to the Input $x^*(t)$ is the sum of the individual impulse responses

$$y(t) = \begin{cases} g(t)x(0) & 0 \le t < T \\ g(t)x(0) + g(t-T)x(T) & T \le t < 2T \\ g(t)x(0) + g(t-T)x(T) + g(t-2T)x(2T) & 2T \le t < 3T \\ g(t)x(0) + g(t-T)x(T) + g(t-2T)x(2T) + \dots + g(t-kT)x(kT) & kT \le t < (k+1)T \end{cases} \longrightarrow (4)$$

$g(t) = 0$ for $t < 0$

$g(t-kT) = 0$ for $t < kT$

so that

$$y(t) = g(t)x(0) + g(t-T)x(T) + g(t-2T)x(2T) + \dots + g(t-kT)x(kT)$$

$$y(t) = \sum_{h=0}^{k} g(t-hT)\,x(hT) \quad 0 \le t \le hT \longrightarrow (5)$$

The values of the output $y(t)$ at the sample instants $t = kT$ $(k = 0,1,2,3 \dots)$ are given by.

$$y(kT) = \sum_{h=0}^{k} g(kT-hT)\,x(hT) \longrightarrow (6)$$

$$= \sum_{h=0}^{k} x(kT-hT)\,g(hT) \longrightarrow (7)$$

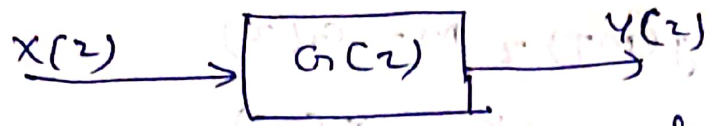
wher $g(kT)$ is the system's weighting sequence. the inverse z-transform of $G(z)$ is called the weighting sequence.

The equations (6) and (7) are called as convolution summation.

The simplified notation is

$$\boxed{y(kT) = x(kT) * g(kT)}$$

Pulse Transfer Function:



The z-transform of the output at the sampling instants to that of sampled input is called as pulse transfer function.

$X(z)$ = Input function

$Y(z)$ = output function.

$G(z)$ = pulse transfer function.

By using the equation of convolution summation.

$$y(KT) = \sum_{h=0}^{\infty} g(KT-hT)\,x(hT), \quad K = 0,1,2,3\ldots$$

where $g(KT-hT) = 0$ for $h > K$.

The z-transform of $y(KT)$ becomes.

$$Y(z) = \sum_{K=0}^{\infty} y(KT)\,z^{-K}$$

$$= \sum_{K=0}^{\infty}\sum_{h=0}^{\infty} g(KT-hT)\,x(hT)\,z^{-K}$$

$$= \sum_{K=0}^{\infty}\sum_{h=0}^{\infty} g[\cancel{KT}(K-h)]\,x(hT)\,z^{-K}$$

$$= \sum_{K=0}^{\infty}\sum_{h=0}^{\infty} g[T(K-h)]\,x(hT)\,z^{-K}$$

$\boxed{\text{Let } m = K-h, \; K = m+h}$, ~~K=0 Means M=~~

$$\cancel{Y(z) = \sum_{m+h} Y(z) =}$$

$$Y(z) = \sum_{K=0}^{\infty}\sum_{h=0}^{\infty} g(mT)\,x(hT)\,z^{-(m+K)}$$

$$= \sum_{m=0}^{\infty}\sum_{h=0}^{\infty} g(mT)\,x(hT)\,z^{-(m+K)}$$

$$= \sum_{m=0}^{\infty}\sum_{h=0}^{\infty} g(mT)\,z^{-m}\,z^{-K}\,x(hT)$$

$$= \sum_{m=0}^{\infty} g(mT)\,z^{-m} \sum_{h=0}^{\infty} x(hT)\,z^{-K}$$

$$Y(z) = G(z)\,X(z)$$

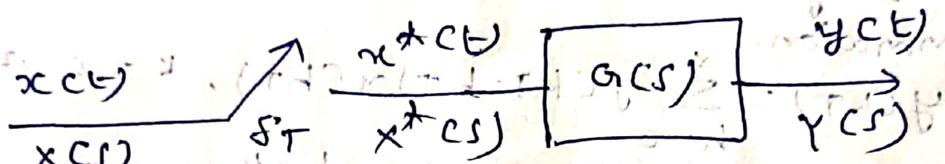
$$\left(\because \sum_{m=0}^{\infty} g(mT)\,z^{-m} = G(z)\right.$$

$$\left.\because \sum_{h=0}^{\infty} x(hT)\,z^{-k} = X(z)\right)$$

Hence $Y(z) = G(z)\,X(z)$

$$\boxed{\therefore\ G(z) = \frac{Y(z)}{X(z)}} \rightarrow \text{pulse transfer function.}$$

Starred Laplace - Transform of a signal:



In analyzing discrete-time control systems, certain signals are Impulse sampled. The pulse transfer function of such signals can be obtained by taking the Laplace transform of the output signal.

The output $Y(s) = G(s)\,X^*(s) \rightarrow (1)$

$G(s) =$ Transfer function.

$Y(s) =$ output function.

$X^*(s) =$ Input signal.

Apply starred laplace transform for the eqn (1).

$$Y^*(s) = [G(s)\,X^*(s)]^*$$

$$= [G(s)]^*\,X^*(s)$$

$$= G^*(s)\,X^*(s) \rightarrow (2)$$

Inverse laplace transform for the equation (1) is.

$$L^{-1}[(Y(s)] = L^{-1}[G(s)\,X^*(s)]$$

$$y(t) = L^{-1}[G(s)\,X^*(s)]$$

$$= \int_0^t g(t-\tau)\,x^*(\tau)\,d\tau$$

$$y(t) = \int_0^t g(t-\tau) \sum_{k=0}^{\alpha} x(\tau)\,\delta(\tau-kT)\,d\tau$$

$$= \sum_{k=0}^{\alpha} \int_0^t g(t-\tau)\,x(\tau)\,\delta(\tau-kT)\,d\tau$$

$$y(t) = \sum_{k=0}^{\alpha} g(t-kT)\,x(kT) \longrightarrow ③$$

The z-transform of $y(t)$ becomes.

$$Y(z) = Z[y(t)] = \sum_{n=0}^{\alpha} \left[\sum_{k=0}^{\alpha} g(nT-kT)\,x(kT)\right] z^{-n}$$

let $n = m+k$

$m = n-k$

$$Y(z) = \sum_{m=0}^{\alpha} \sum_{k=0}^{\alpha} g(mT)\,x(kT)\,z^{-(k+m)}$$

$$\therefore Y(z) = \sum_{m=0}^{\alpha} \left[\sum_{k=0}^{\alpha} g(T(n-k))\,x(kT)\right] z^{-(m+k)}$$

$$Y(z) = \sum_{m=0}^{\alpha} g(mT)\,z^{-m} \sum_{k=0}^{\alpha} x(kT)\,z^{-k}$$

$$Y(z) = G(z)\,X(z) \longrightarrow ④$$

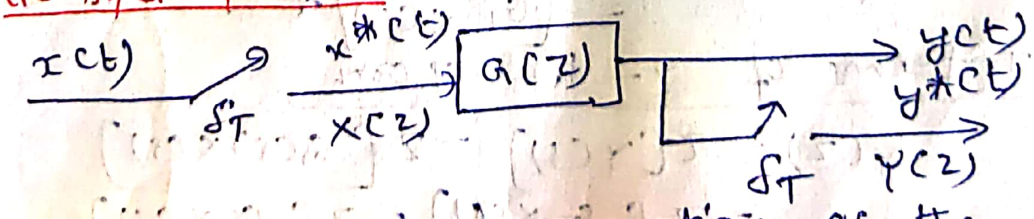
The starred Laplace transform can be obtained by replacing z with e^{TS}.

The equation ④ become.

$$Y^*(s) = G^*(s)\,X^*(s)$$

$$\boxed{Y^*(s) = G^*(s)\,X^*(s)}$$

General procedures for obtaing pulse-transfer function:



The pulse transfer function of the above system is

$$\boxed{\frac{Y(z)}{X(z)} = G(z) = Z[G(s)]} \longrightarrow ①$$

$$\xrightarrow[X(s)]{x(t)} \boxed{G(s)} \xrightarrow[Y(s)]{y(t)}$$

Now the transfer function of above system is $\boxed{\dfrac{Y(s)}{X(s)} = G(s)}$

note that the pulse transfer function for this system is not $Z[G(s)]$, because of the absence of the input sampler.

Laplace transform of $y(t)$ is

$$Y(s) = G(s)\, X(s)$$

by taking the starred Laplace transform of $Y(s)$, we have.

$$Y^*(s) = [G(s)\, X(s)]^*$$

$$\boxed{Y^*(s) = G^*(s)\, X^*(s)}$$

or interms of z- transform.

$$\cancel{Y(z) = Y^*(s) = [G\, X(s)]^*}$$

$$\cancel{Y(z) = Z[Y(s)]}$$

$$* \quad \boxed{Y(z) = G(z)\, X(z)} \longrightarrow (2)$$

For Fig (b) the Laplace transform of the output $y(t)$ is.

$$Y(s) = G(s)\, X(s)$$

$$Y^*(s) = [G(s)\, X(s)]^*$$

$$= [G\, X(s)]^*$$

Interms of z- transform.

$$Y(z) = Z[Y(s)] = Z[G(s)\, X(s)]$$

$$= Z[G\, X(s)] = Z[G\, X(s)]$$

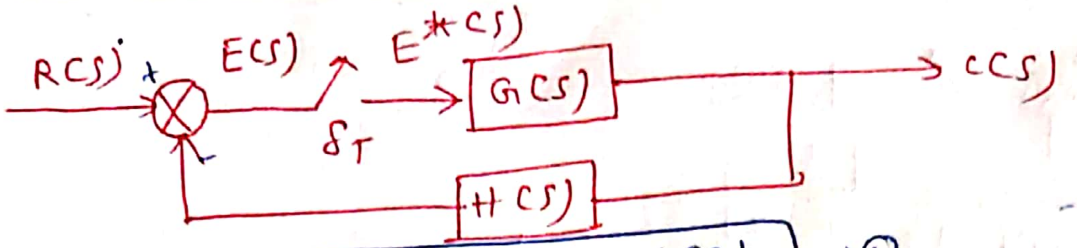
$$\boxed{Y(z) = G\, X(z)} \longrightarrow (3)$$

by comparing equations (2) and (3)

$$\boxed{G\, X(z) \neq G(z)\, X(z)}$$

pulse-transfer function of closed-loop system

1.



$$\boxed{E(s) = R(s) - H(s)\,C(s)} \rightarrow ①$$

$$C(s) = G(s)\,E^*(s) \rightarrow ②$$

$$E(s) = R(s) - H(s)\,G(s)\,E^*(s)$$

by taking starred-Laplace transform.

$$E^*(s) = R^*(s) - GH^*(s)\,E^*(s)$$

$$E^*(s) + GH^*(s)\,E^*(s) = R^*(s)$$

$$E^*(s)\left[1 + GH^*(s)\right] = R^*(s)$$

$$\boxed{E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}} \rightarrow ③$$

Apply starred Laplace-transform for the eqn ②

$$C^*(s) = G^*(s)\,E^*(s)$$

$$C^*(s) = \frac{G^*(s)\,R^*(s)}{1 + GH^*(s)}$$

Interms of z-transform.

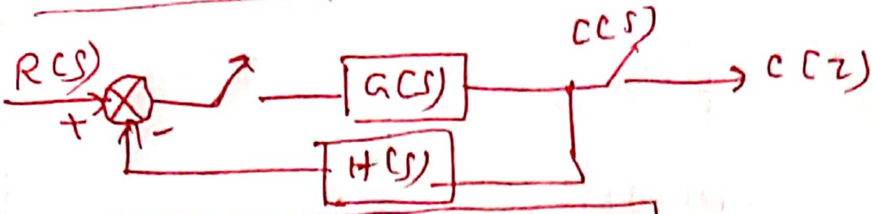
$$C(z) = \frac{G(z)\,R(z)}{1 + GH(z)} \rightarrow ④$$

The inverse z-transform of the above equation gives the values of the output at the sampling instants.

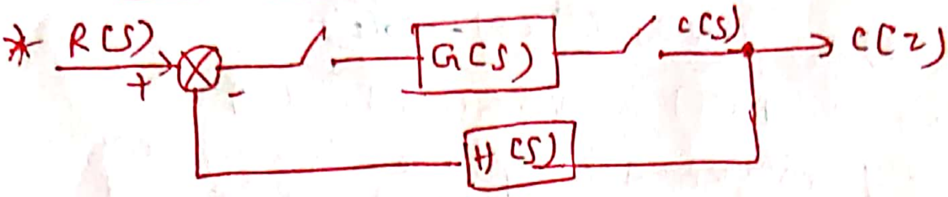
The pulse transfer function for the present closed loop system is

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}}$$

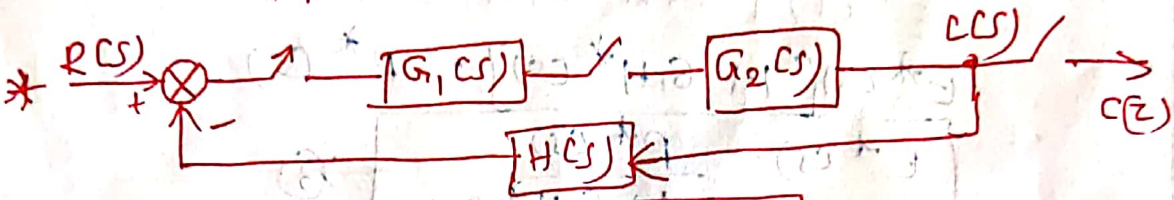
Typical configurations:



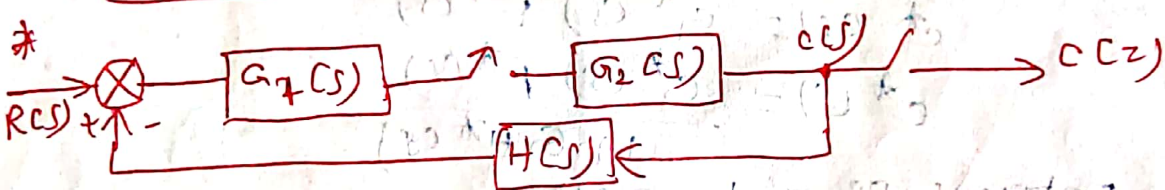
$$\boxed{C(z) = \frac{R(z)\,G(z)}{1+GH(z)}}$$

* 

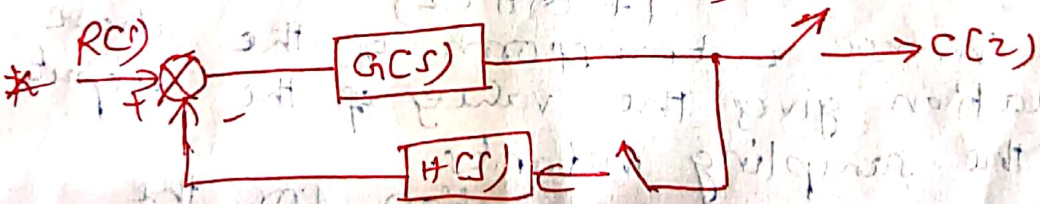
$$C(z) = \frac{R(z)\,G(z)}{1+G(z)\,H(z)}$$

* 

$$\boxed{C(z) = \frac{R(z)\,G_1(z)\,G_2(z)}{1+G_1(z)\,G_2H(z)}}$$

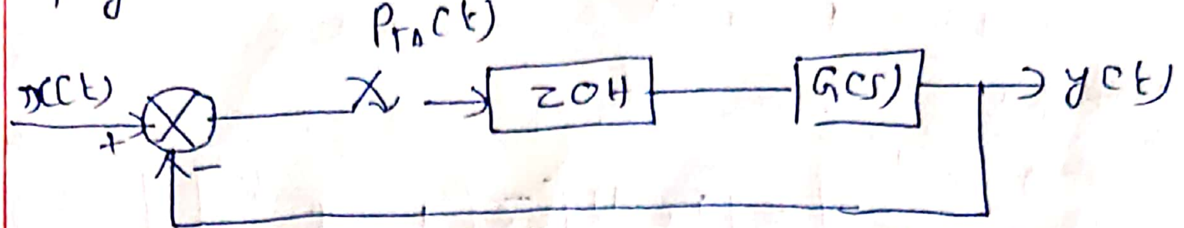
* 

$$C(z) = \frac{\cancel{R(z)}\; G_1R(z)\,G_2(z)}{1+G_1G_2H(z)}$$

* 

$$\boxed{C(z) = \frac{GR(z)}{1+GH(z)}}$$

b) Write the difference equation governing the system for $G(s) = \frac{1}{s+1}$ as shown in figure.

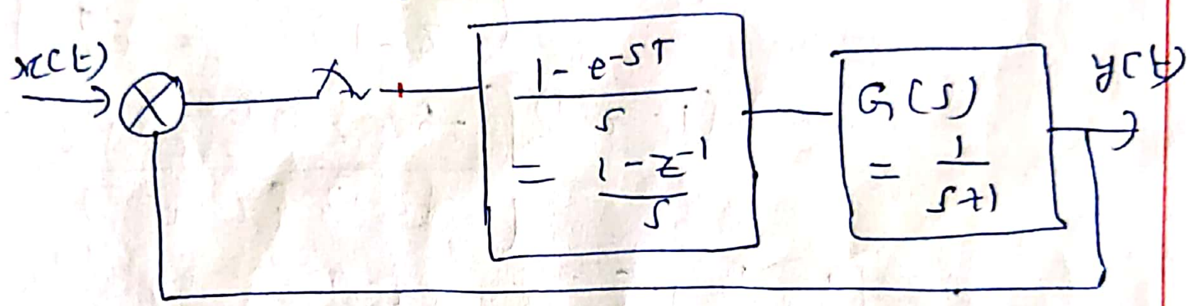


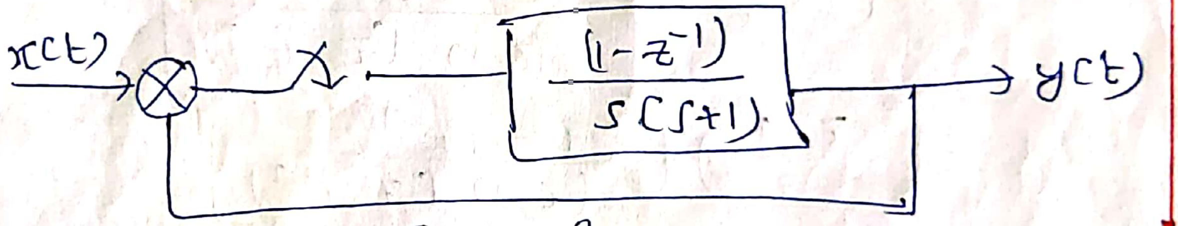
A: clearly : $H(s) = 1$

we know that T.F of ZOH $= \frac{1-e^{-sT}}{s}$

$(\because e^{+sT} = z)$

$e^{-sT} = z^{-1}$





$$\frac{Y(z)}{X(z)} = \frac{Z[G_e(s)]}{1+Z[G_e H(s)]} \quad (\because G_e = G_{OH}G)$$

$$\frac{Y(z)}{X(z)} = \frac{Z\left[\frac{1-z^{-1}}{s(s+1)}\right]}{1+Z\left[\frac{1-z^{-1}}{s(s+1)}\right]} \rightarrow (1)$$

$$Z\left[\frac{1-z^{-1}}{s(s+1)}\right] = (1-z^{-1})\, Z\left[\frac{1}{s(s+1)}\right] \rightarrow (2)$$

let

$$F(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \rightarrow (3)$$

$$A = \left.\frac{1}{s+1}\right|_{s=0} = 1$$

$$B = \left.\frac{1}{s}\right|_{s=-1} = -1$$

From ③:

$$F(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$f(t) = 1 - e^{-1t}$$

From ②:

$$Z\left[\frac{1-z^{-1}}{s(s+1)}\right] = (1-z^{-1})\left[Z\left[1 - e^{-t}\right]\right]$$

$$= (1-z^{-1})\left[Z(1) - Z\left[e^{-t}\right]\right]$$

$$= (1-z^{-1})\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right)$$

$$= (1-z^{-1})\left[\frac{z^2 - ze^{-T} - z^2 + z}{(z-1)(z-e^{-T})}\right]$$

$$= \left(1-\frac{1}{z}\right)\left(\frac{z - ze^{-T}}{(z-1)(z-e^{-T})}\right]$$

$$= \left(\frac{z-1}{z}\right)\left(\frac{z - ze^{-T}}{(z-1)(z-e^{-T})}\right]$$

$$= \left(\frac{z-1}{z}\right)\left(\frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}\right]$$

$$\boxed{\frac{Y(z)}{X(z)} = \frac{1-e^{-T}}{z-e^{-T}}}$$

From ①:

$$\frac{Y(z)}{X(z)} = \frac{\frac{1-e^{-T}}{z-e^{-T}}}{1+\frac{1-e^{-T}}{z-e^{-T}}}$$

$$\frac{Y(z)}{X(z)} = \frac{1-e^{-T}}{z-e^{-T}+1-e^{-T}} = \frac{1-e^{-T}}{z+1-2e^{-T}}$$

$$\boxed{\frac{Y(z)}{X(z)} = \frac{1-e^{-T}}{z+1-2e^{-T}}}$$

For Differential equation

$$Y(z)\,z + Y(z) - 2Y(z)e^{-T} = X(z) - X(z)e^{-T}$$

Apply Inverse z transform.

$$z^{-1}[zY(z)] + z^{-1}[Y(z)] - z^{-1}[2Y(z)e^{-T}] = z^{-1}[X(z)] - z^{-1}[X(z)e^{-T}]$$

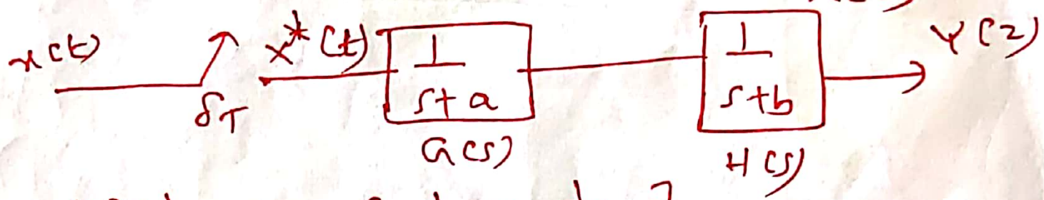
$$y(k+1) + y(k) - 2e^{-T}y(k) = x(k) - e^{-T}x(k)$$

*

$$\boxed{y(k+1) + y(k)(1-2e^{-T}) = x(k)(1-e^{-T})}$$

↳ Differential equation.

* obtain pulse transfer function $\frac{Y(z)}{X(z)}$



$$\frac{Y(z)}{X(z)} = Z\left[\frac{1}{s+a} \cdot \frac{1}{s+b}\right]$$

$$= Z\left[\cancel{s}\ \frac{1}{(s+a)(s+b)}\right] \rightarrow (1)$$

Let $F(z) = \frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$

$$A = \left.\frac{1}{s+b}\right|_{s=-a} = \frac{1}{-a+b}$$

$$B = \left.\frac{1}{s+a}\right|_{s=-b} = \frac{1}{-b+a}$$

$$F(z) = \frac{1}{(-a+b)(s+a)} + \frac{\cancel{-b+a}\ 1}{(-b+a)(s+b)}$$

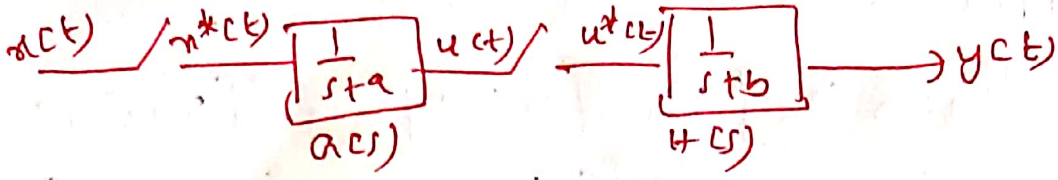
$$f(t) = \frac{1}{(-a+b)} e^{-at} + \frac{1}{(-b+a)} e^{-bt}$$

NOW

$$\frac{Y(z)}{X(z)} = Z\left(\frac{e^{-at}}{(-a+b)} + \frac{e^{-bt}}{a-b}\right)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{b-a}\left(\frac{z}{z-e^{-aT}}\right) + \left(\frac{1}{a-b}\right)\left(\frac{z}{z-e^{-bT}}\right)$$

$$\boxed{\frac{Y(z)}{X(z)} = GH(z) = \frac{1}{b-a}\left(\frac{z}{z-e^{-aT}} - \frac{z}{z-e^{-bT}}\right)}$$



$$\frac{Y(z)}{X(z)} = Z[G(s)H(s)]$$

$$= G(z)H(z)$$

$$= Z\left[\frac{1}{s+a}\right] \cdot Z\left[\frac{1}{s+b}\right]$$

$$= Z[e^{-at}] \cdot Z[e^{-bt}]$$

$$\boxed{\frac{Y(z)}{X(z)} = \left(\frac{z}{z-e^{-aT}}\right)\left(\frac{z}{z-e^{-bT}}\right)}$$