UNIT-1
Introduction and signal processing

Syllabus: Introduction to analog and digital control systems- Advantages of digital systems - Typical examples - signals and processing - sample and hold devices - sampling theorem and data reconstruction - Frequency domain characteristics of zero order hold.

Introduction: A control system is an interconnection of components to provide a desired function.

Automatic control systems play a vital role in the (technological) progress of human civilization.

From the 1980's onwards, we find microprocessor digital technology to be a dominant industrial phenomenon. To day, the most complex industrial processes are under computer control.

The current trend toward digital rather than analog control of dynamic systems is mainly due to the availability of low-cost digital computers and the advantages found in working with digital systems rather than continuous time signals.

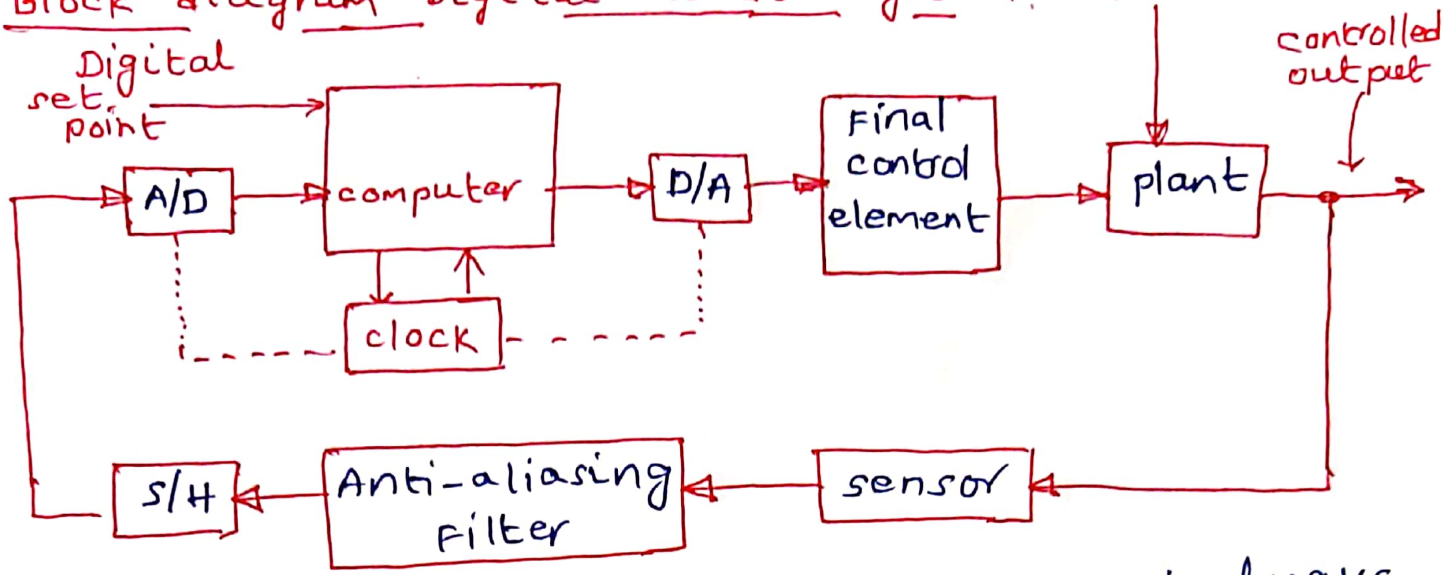
Analog control systems: The controller in an analog control system is made up of analog components like resistor, capacitors and operational amplifiers.

Digital control systems: It is made up of digital hardware. The heart of the Digital control system is digital controller i.e computer usually in the form of programmed digital computer.

Digital controllers normally have analog devices at their periphery to interface with the plant.

Digital control systems are also called computer control systems.

Block diagram Digital control system:

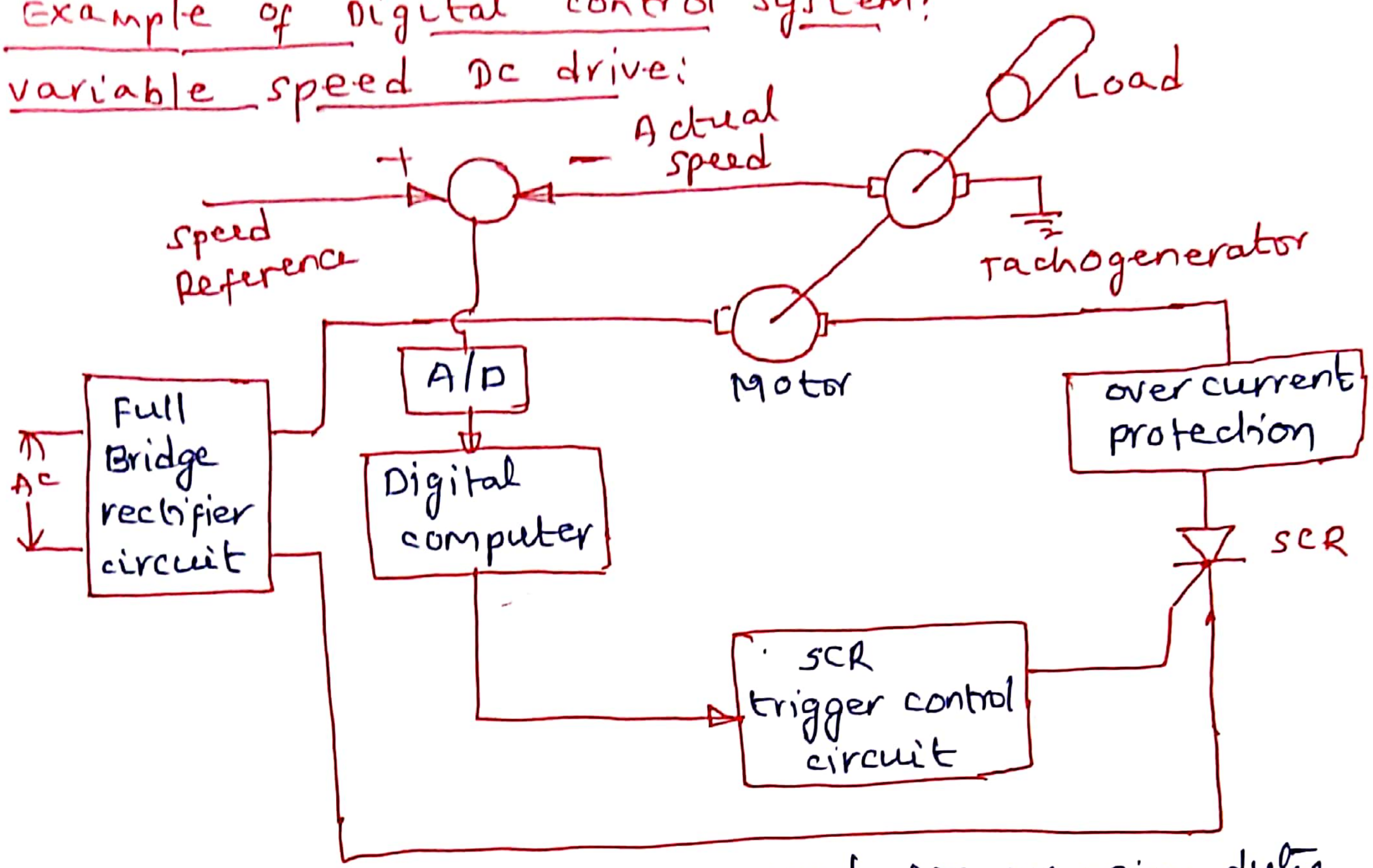


The input and output of the system is always an analog signals.

* The output of the plant is a controlled output. This output is fed back to input through feedback. i,e sensor. It is an analog signal.
* The analog Feedback signal coming From the sensor is usually of low Frequency and consists high frequency noise. This noise can be eliminated by using anti-alising filter.
* The analog signal, after anti-aliasing processing, is converted into digital form by the Analog to digital conversion system. It consists sample and hold devices.
* The digital computer processes the series of numbers by using an ~~algroth~~ algorithm and generates a new series of numbers.
* The actuator (Final control element) accepts only an analog input. so that this new series of numbers must be converted into a continuous time or analog signal by Digital to analog converter. The resulting analog signal is applied to the actuator to control the plants behaviour. The real time clock in the computer, synchronizes all the events of A/D conversion and D/A conversion.

Example of Digital control system:

variable speed DC drive:



* variable speed drives are used for pumping duty to vary the flow rate or the pumping pressure, rolling mills, rail traction etc..
* The SCR are almost universally used to control the speed of dc motors. Figure shows dc motor driven by a full-wave rectified supply. Armature current of the dc motor is controlled by an SCR, which is, in turn, controlled by the pulses applied by the SCR trigger control circuit. The SCR controller thus combines the functions of a D/A converter and final control element.
* Firing angle of the SCR controls the average armature current, which, in turn, controls the speed of the dc motor.
* The reference voltage which corresponds to the desired speed of the dc motor, is compared with the output voltage of tachogenerator, corresponding to the actual speed of the motor. The error signal is pass through the A/D and the digital computer

i,e controller. The digital controller controls the firing angle generated by the trigger circuit and finally the desired speed will be achieved.

Advantages of Digital control systems:

1) **Flexibility:** The ability to "redesign" the controller by changing software is an important feature of digital control against analog control.

2) **speed:** Due to high speed Digital computers, it is possible to sample and process control signals at very high speed.

3) **Accuracy:** Digital signal offers very small error in comparision with analog signal.. Analog signals are more noise as compared to Digital signals.

4) **Future Generation control systems:** The Future generation control systems will have a significant Artificial Intelligence component, the list of applications of computer-based control will continue to expand.

5) **Integrated control of Industrial systems:** Real time applications of information processing and decision-making e.g, production planning, scheduling, optimization, operations control, etc.. may now integrated with the traditional process control functions.

6) **cost:** The cost of digital circuits used in digital controller is very low.

Disadvantages:

1) **Quantization effect:** Here we use ADC and DAC For signal conversion. In this process precious amount of information relating to the signal is lost.

2) **sampling Effects:** The process of sampling and reconstruction also affects the amount of information

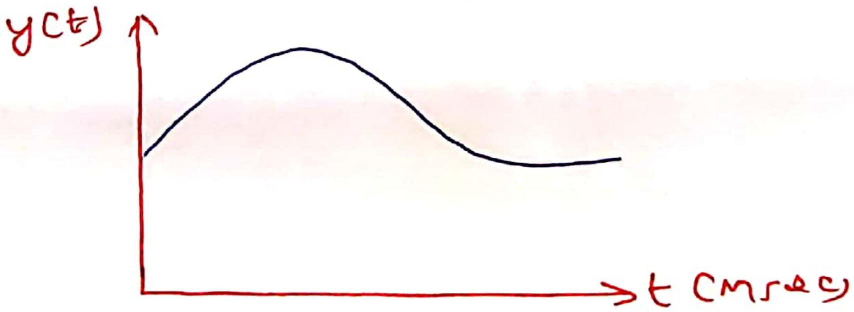
available to the control computer, and degrades control system performance. For example, converting a given continuous time control system into a digital control system, without changing the system parameters, degrades the system stability margin.

Note: with the availability of low-cost, high performance digital computers and interfacing hardware, the implementation problems in digital control do not pose a serious threat to its usefulness. The advantages are most dominant the disadvantages.

signals and processing:

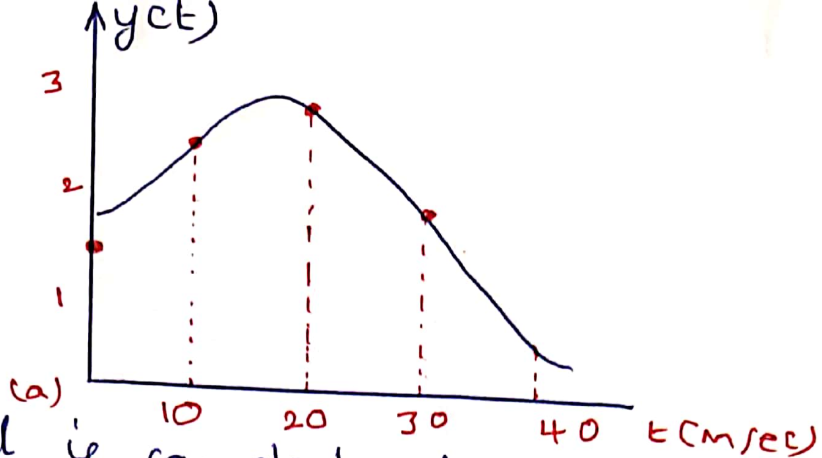
continuous Time signal: A continuous-time signal is a signal defined over a continuous range of time. The amplitude may assume a continuous range of values or may assume only a finite no. of distinct values.

An analog signal y(t) - it is defined at the continuous of times.

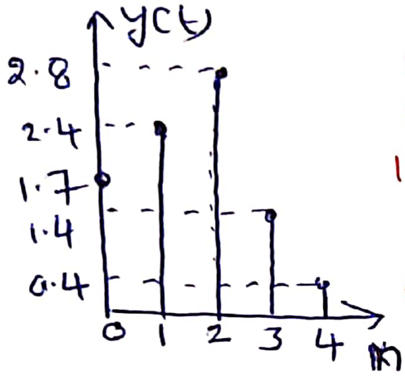


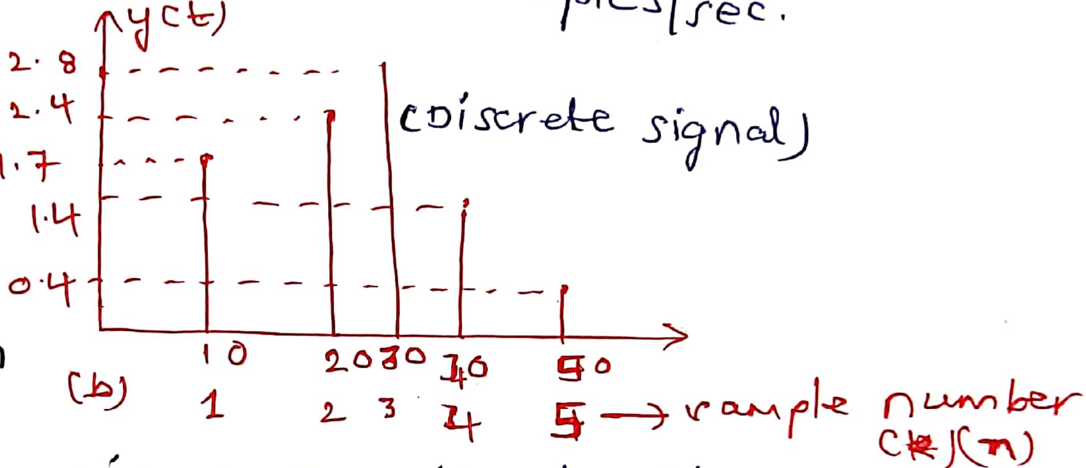
such a signal cannot be stored in digital computers. Therefore it must be converted to a form that will be accepted by digital computers. One very common method to do this is to record sample values of this signal at equally spaced instants.

E.g: consider an analog signal y(t) and we sample the signal every 10msec.



This signal is sampled at every 10msec. This signal is called <u>discrete-time signal</u>. The sampling interval of 10msec corresponds to a sample rate of 100 samples/sec.





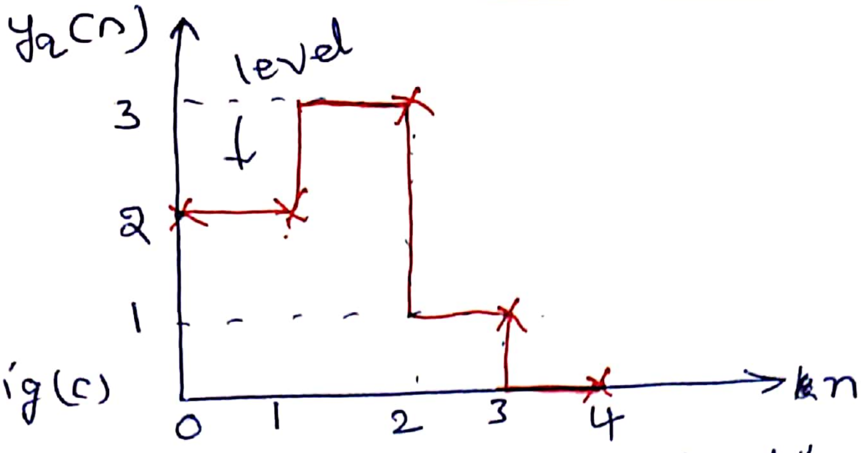
The time axis of the discrete-time signal is labeled simply "sample number" (n = 0,1,2...). corresponding to different values of sample number n, the discrete-time signal assumes the same continuous range of values assumed by the analog signal y(t).
we can represent the samples by a sequence of numbers ys.

$$ys = \{1.7, 2.4, 2.8, 1.4, 0.4 \ldots)$$

In general $ys = \{y(n)\}$, ~~0≤n≤α~~ $0 \le n < \alpha$

where $y(n)$ denotes the nth number in the sequence.

" we can assume that any value in the interval [0.5, 1.5] is rounded to 1 and it is called quantization.



Fig(c)

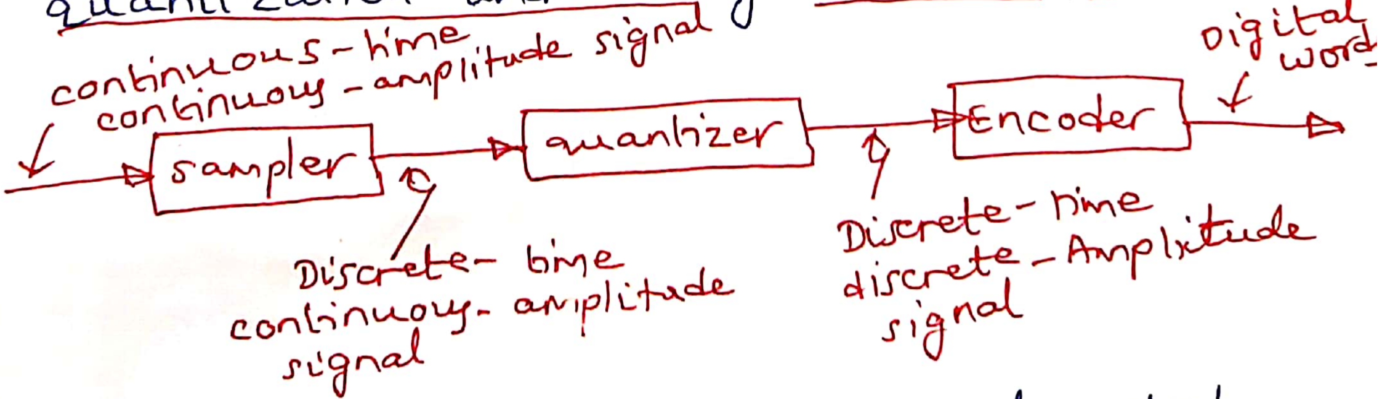
After sampling and quantization, the final step required in converting an analog signal to a form acceptable to digital computer is coding (or encoding). The encoder maps each quantized sample value into a digital word.

n	Digital word
0	10
1	10
2	11
~~4~~ 3	10
4	00

Fig (d)

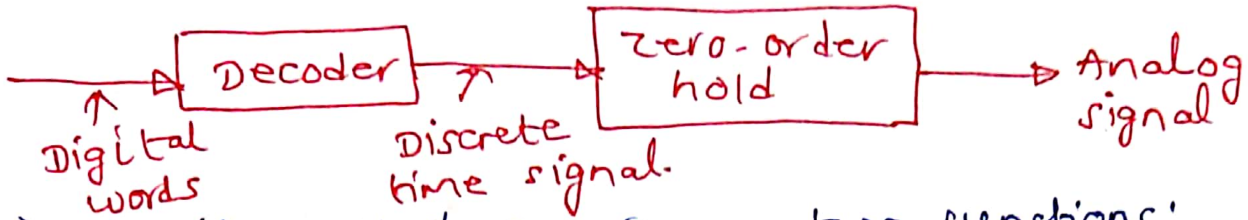
Fig (d) gives the coded signal, corresponding to the analog signal of fig (a)

"The device that performs the sampling, quantization and coding is an A/D converter."

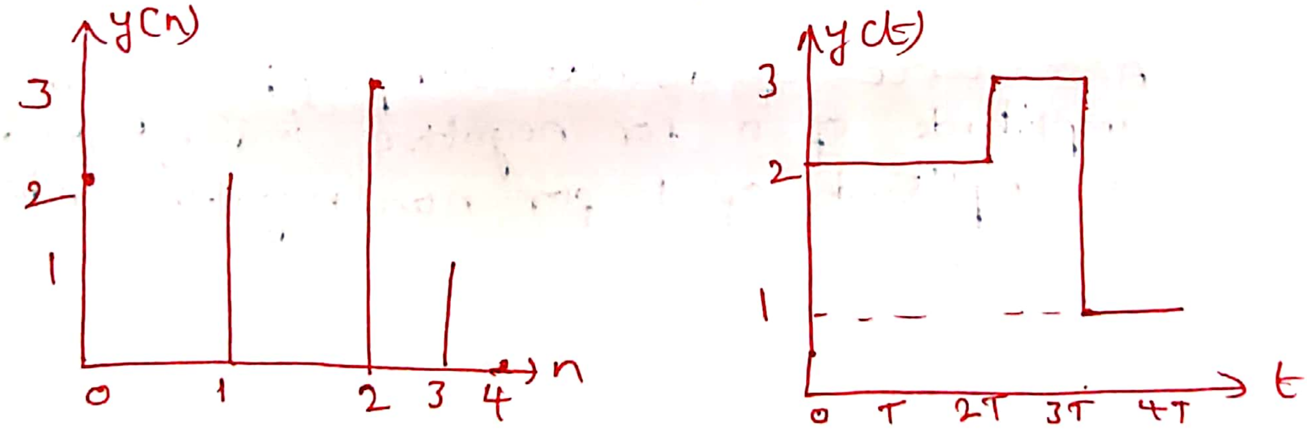


It has been assumed that the final control element (actuator) is an analog component. So we need to ~~x~~ convert digital signal into an analog signal.

operations performed by a D/A converter

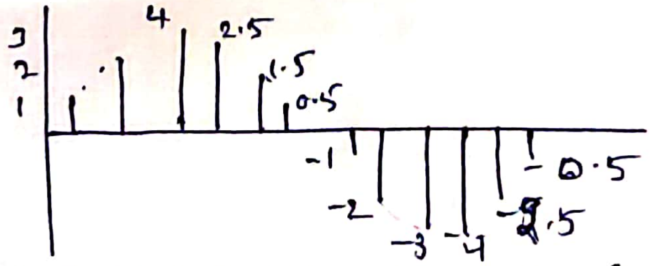


* The D/A converter performs two functions: First generation of output samples from the binary form. The decoder maps each digital word into a sample value of the signal in discrete-time form.
* second, conversion of these samples to analog form. The simplest way of converting a sample sequence into a continuous-time signal is to hold the value of the sample until the next one arrives. The net effect is to convert a sample to a pulse of duration T – the sample period. This operation of a D/A converter is referred to as a zero-order Hold (ZOH) operation.



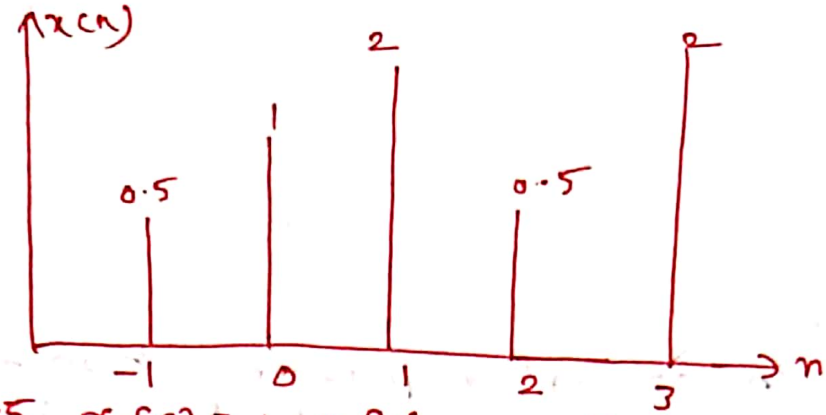
Basic Discrete-Time signals:

The signals which are discrete in time but continuous in amplitude are called discrete-Time signals.



$x[n] = \{1, 2, 3, 4, 2.5, 1.5, 0.5, -1, -2, -3, -4, -2.5, -0.5\}$

Eg:



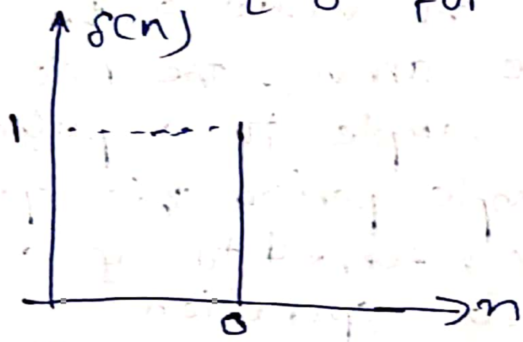
$x(-1) = 0.5$, $x(0) = 1$, $x[1] = 2$, $x[2] = 0.5$

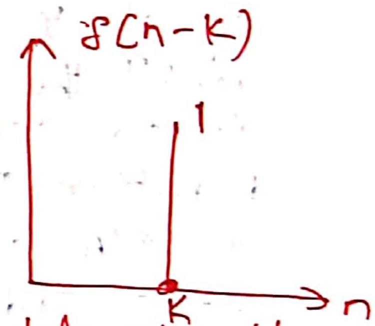
$x[3] = 2.$

$x(n) = \{0.5, 1, 2, 0.5, 2\}$ $n = -1, 0, 1, 2$ and 3

1. unit Impulse sequence: A sequence of discrete samples having an amplitude of 1 at origin and an amplitude of 0 for all other instants

$$\delta[n] = \begin{cases} 1, & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



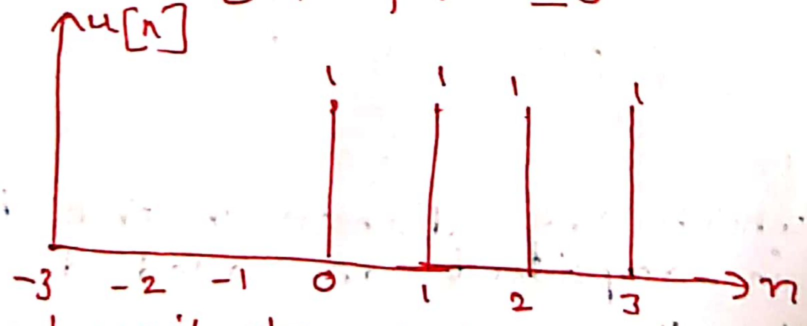


delayed unit sample sequence

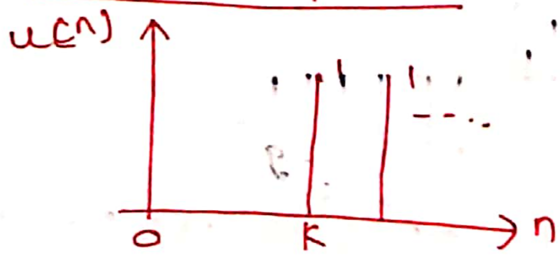
2) unit-step sequence:

A sequence of discrete samples having an amplitude of 0 for negative indices and an amplitude of 1 for non-negative indices.

$$u[n] = \begin{cases} 0, & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$



Delayed unit-step sequence:



$$u[n-K] = \begin{cases} 1 & \text{for } n \geq K \\ 0 & \text{for } n < K \end{cases}$$

3) sinusoidal sequence:

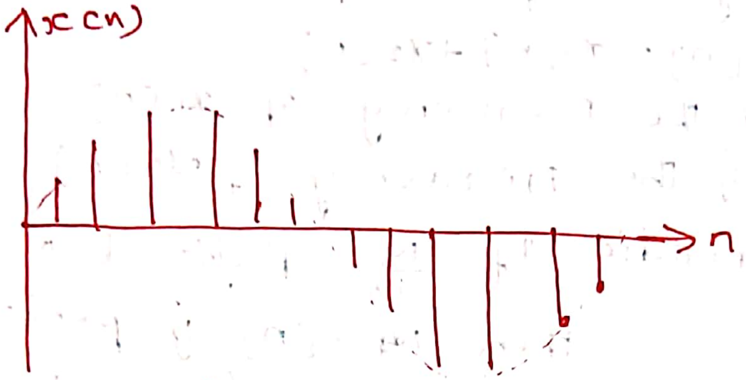
$$x[n] = A \cos(\omega_0 n + \phi)$$

(or)

$$x[n] = A \cos(\Omega n + \phi)$$

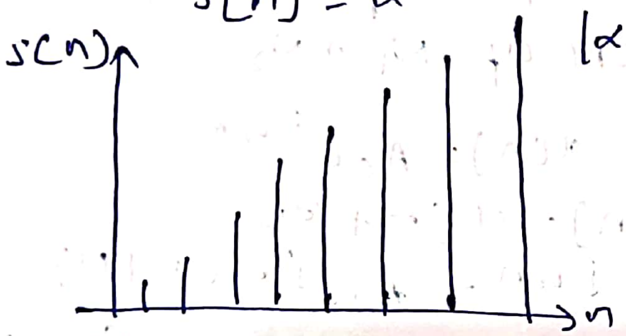
$\omega_0 = \Omega =$ Frequency

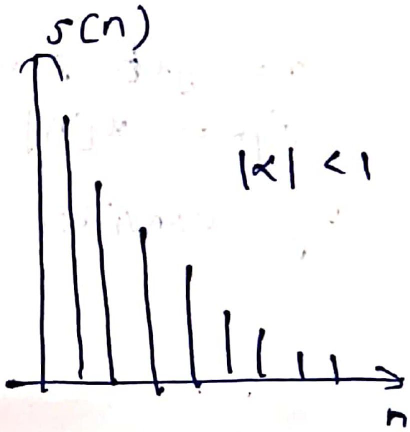
$\phi =$ phase.



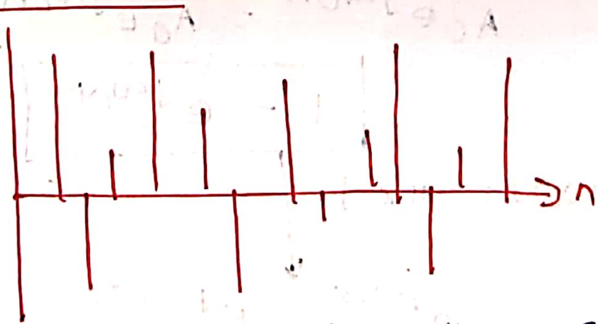
4) Exponential sequence: It represents exponentially increasing or decreasing series.

$$s[n] = \alpha^n$$



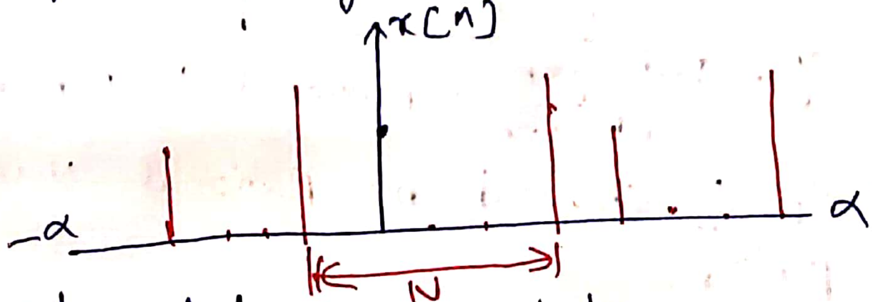


5) Random sequence:



periodic & Aperiodic Discrete Time signals:

A discrete time signal $x[n]$ is said to be periodic if it repeats at regular intervals of time.



N = Fundamental time periode

<u>condition:</u> $x[n] = x[n \pm kN]$

$+kN$ for left shift

$-kN$ for Right shift

k = Integer

A signal becomes Aperiodic if $x[n] \neq x[n \pm kN]$

<u>For composite signal:</u> If A signal

$x[n] = x_1[n] + x_2[n]$

N_1 is the Frequency of $x_1[n]$

N_2 is the Frequency of $x_2[n]$

$x[n]$ is now periodic If $\frac{N_1}{N_2}$ = Rational Number

then ~~N~~ $x[n]$ is periodic

If $\frac{N_1}{N_2} \neq$ Irrational Number then $x[n]$, is Aperiodic.

To get fundamental Time periode of composite signal $x[n]$ Take LCM of , N_1 & N_2.

<u>Eg:</u> consider a signal $x[n] = A_0 e^{j\omega_0 n}$

$$x[n] = x[n + kN]$$

$$A_0 e^{j\omega_0 n} = A_0 e^{j\omega_0 (n+N)}$$

$$[\because k = 1]$$

$$A_0 e^{j\omega_0 n} = A_0 e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

$$\boxed{1 = e^{j\omega_0 N}}$$

$(\because \omega_0 N = 2\pi k)$

$[\because 1 = e^{j\omega}$
$1 = e^{j2\pi k}]$
for any integer k]

$$e^{j2\pi k} = e^{j\omega_0 N}$$

$$2\pi k = \omega_0 N$$

$$\boxed{\frac{2\pi}{\omega_0} = \frac{N}{k}}$$

$$\frac{N}{k} = \frac{\text{Integer}}{\text{Integer}} = \text{Rational Number}$$

The Ratio of $\frac{N}{k}$ should be Rational Number to become periodic problem: $x[n] = e^{j2n}$

condition $\frac{2\pi}{\omega_0}$ = Rational Number

$A_0 e^{j\omega_0 n} = e^{j2n}$

$\omega_0 = 2$

$\frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$ It is irrational number

hence the given signal $x[n] = e^{j2n}$ is a Aperiodic signal. It does not exist a period N.

2) $x[n] = \cos[\frac{3\pi}{4} n]$

Here $\omega_0 = \frac{3\pi}{4}$

condition $\frac{2\pi}{\omega_0}$ = Rational number

$\frac{2\pi}{\omega_0} = \frac{2\pi}{3\pi/4} = \frac{8}{3}$ (Rational number)

Hence the given signal $x[n] = \cos[\frac{3\pi}{4} n]$ is a periodic signal.

$\frac{2\pi}{\omega_0} = \frac{N}{K}$

$N = \frac{2\pi}{\omega_0} K = \frac{8}{3} \times K$

note that N is an integer. so To get an integer value K must be $= 3$

so Fundamental periode $N = \frac{8}{3} \times 3$

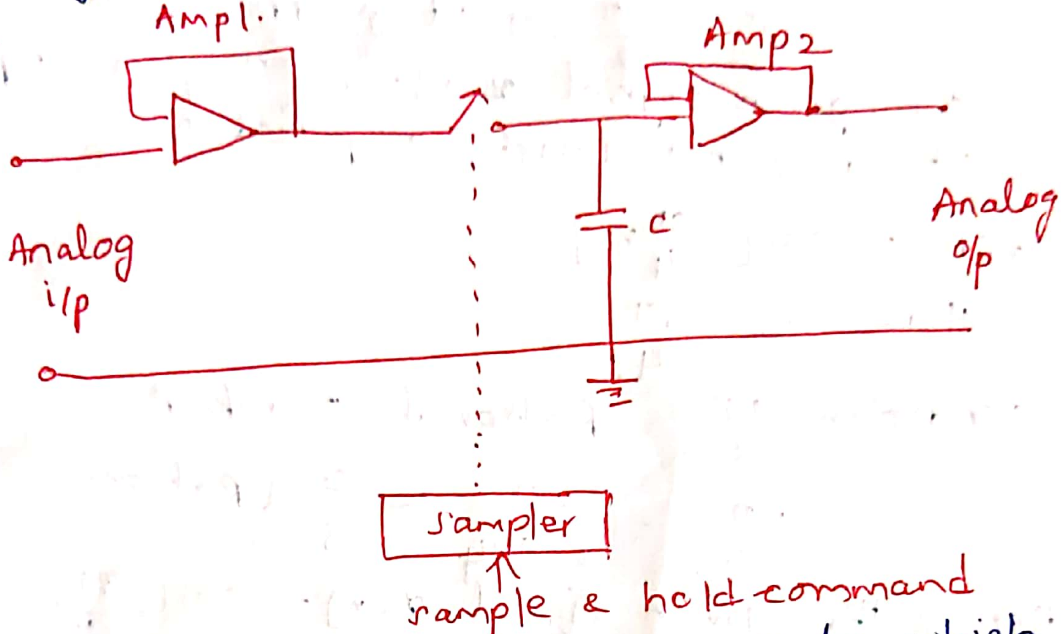
$\boxed{N = 8}$

(∴ N is the Fundamental periode so that K must be minimum)

sampling

sample and hold circuit: A sampler in a digital system converts an analog signal into a train of Amplitude modulated pulses.
The hold circuit holds the value of the sampled pulses signal over a specied period of time.

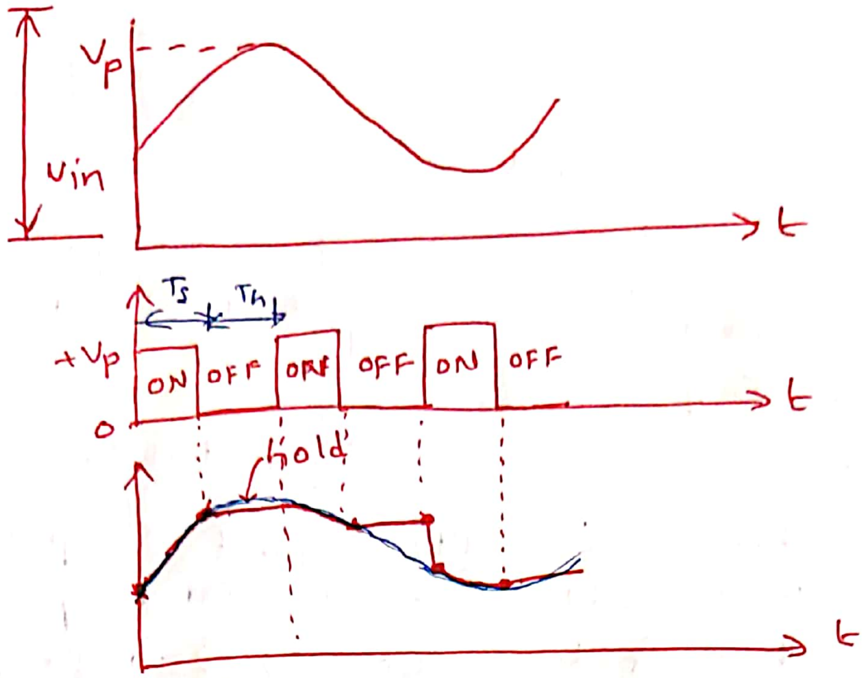
sample and Hold circuit samples the input signal and holds on to its last sampled value until the input is sampled again.

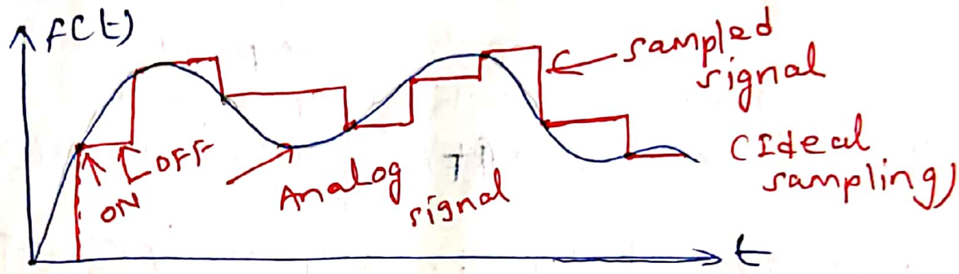


~~Wor~~ working: The Main components which a sample and hold circuit involves is a switch (It May be a MOSFET), a capacitor to store and hold the electric charge and operational amplifier.

When the switch is ON, by a sample command, then the analog signal applied to it will be fed to the capacitor. The capacitor will then charge to its peak value.

When the switch is off, then the capacitor stops charging. Due to the high impedance operational amplifier connected at the end of the circuit, the capacitor will experience high impedance due to this its cannot get discharged. This leads to the holding of the charge by the capacitor for the definite amount of time. This time can be referred as holding period. And the time in which samples of the input voltage is generated is called sampling period.



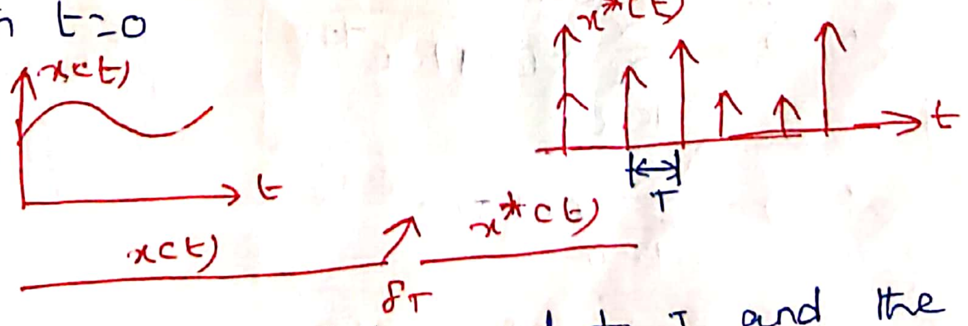


Acquisition Time: The time required by the capacitor to get the charge of the input voltage applied to the sample and hold circuit.

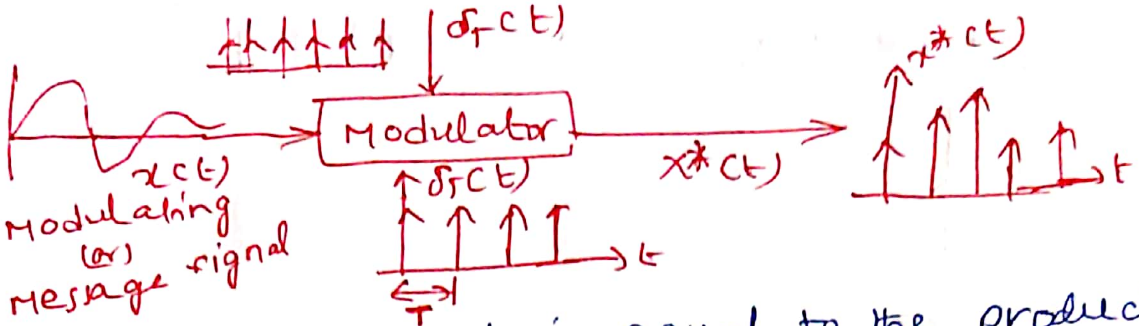
Aperture Time: Time required by the capacitor to change its state from sampling to holding.

Impulse sampler: (Ideal sampler): Modelling:

The output of this sampler is considered to be a train of impulses that begins with t=0



The sampling period equal to T and the strength of each impulse equal to the sampled value of the continuous time signal at the corresponding sampling instant.



The sampler output is equal to the product of the continuous-time input $x(t)$ and the train of Impulses $\delta_T(t)$

$$\text{Impulse Train} = \delta_T(t) = \sum_{k=0}^{\infty} \delta(t-kT)$$

at $k=0$, then $\delta_T(t) = \delta(t) \rightarrow$ First sample
$k=1$ then $\delta_T(t) = \delta(t-T) \rightarrow 2^{nd}$ sample
$k=2$ then $\delta_T(t) = \delta(t-2T) \rightarrow 3^{rd}$ sample

Now $x^*(t)$ = Impulse sampled signal
$x(t)$ = continuous signal

then $x^*(t) = x(\cancel{kT}) \sum_{k=0}^{\infty} \delta(t-kT) \rightarrow ①$

[∴ at time $t=kT$, the Impulse is $x(kT)\,\delta(t-kT)$]

Expand equation ①:

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \dots + x(kT)\delta(t-kT) + \dots \rightarrow ②$$

Apply Laplace Transform

$$L\{x^*(t)\} = L\{x(0)\delta(t)\} + L\{x(T)\delta(t-T)\} + L\{x(2T)\delta(t-2T)\} + \dots + L\{x(kT)\delta(t-kT)\}$$

$$X^*(s) = L\left\{\sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)\right\}$$

$$\boxed{X^*(s) = \sum_{k=0}^{\infty} x(kT)\, e^{kTs}}$$

$$\boxed{X^*(s) = \sum_{k=0}^{\infty} x(kT)\, e^{-kTs}}$$

[∴ $L\{\delta(t)\} = 1$
$L\{\delta(t-kT)\} = e^{-Ts}$]
e^{-kTs}]

Here s is a complex variable $s = \sigma + j\omega$ and its not a rational.

so take Mapping from s-plane to z plane

take $e^{TS} = z$

$$x^*(s) = \sum_{k=0}^{\infty} x(kT)\, e^{-kTs}$$

$$\boxed{X(z) = x^*(s) = \sum_{k=0}^{\infty} x(kT)\, z^{-k}}$$

↳ This is the basic definition of z-transform.

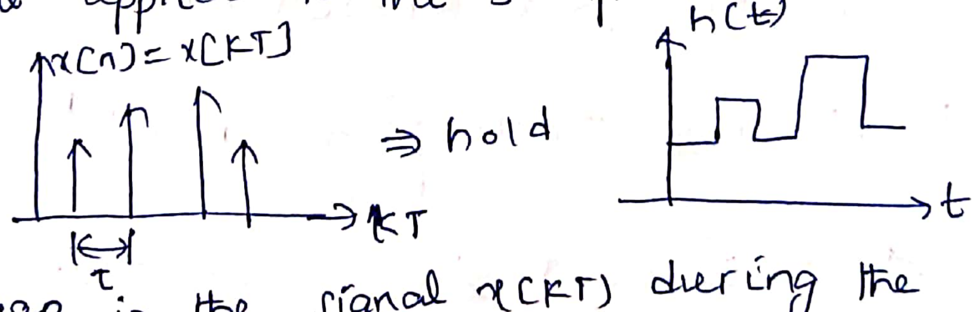
clearly the Laplace transform of Impulse sampled signal $x^*(t)$ is equal to the z-transform of the continuous time signal

$$\boxed{L.T[x^*(t)] = Z.T[x(t)]}$$

switch is a z transform.

$$\boxed{x(t) \xrightarrow[z]{} X(z)}$$

Data-hold circuits: Data hold is a process of generating a continuous-time signal $h(t)$ from a discrete time sequence $x[kT]$ or $x[n]$. A ~~hold~~ hold circuit converts the sampled signal into a continuous time signal. which approximately reproduces the signal applied to the sampler.



The gap in the signal $x(kT)$ during the time interval $kT \le t \le (k+1)T$ is estimated by a polynomial in 'τ' and produces $h(t)$

$$h(kT+\tau) = a_n \tau^n + a_{n-1} \tau^{n-1} + \ldots + a_1 \tau + a_0 \quad \text{↳①}$$

The signal $h(kT)$ must equal $x(kT)$ or

$$h(kT) = x(kT)$$

The equation ① can be rewritten as

$$h(kT+t) = a_n \tau^n + a_{n-1} \tau^{n-1} + \ldots + a_1 \tau + x(kT)$$

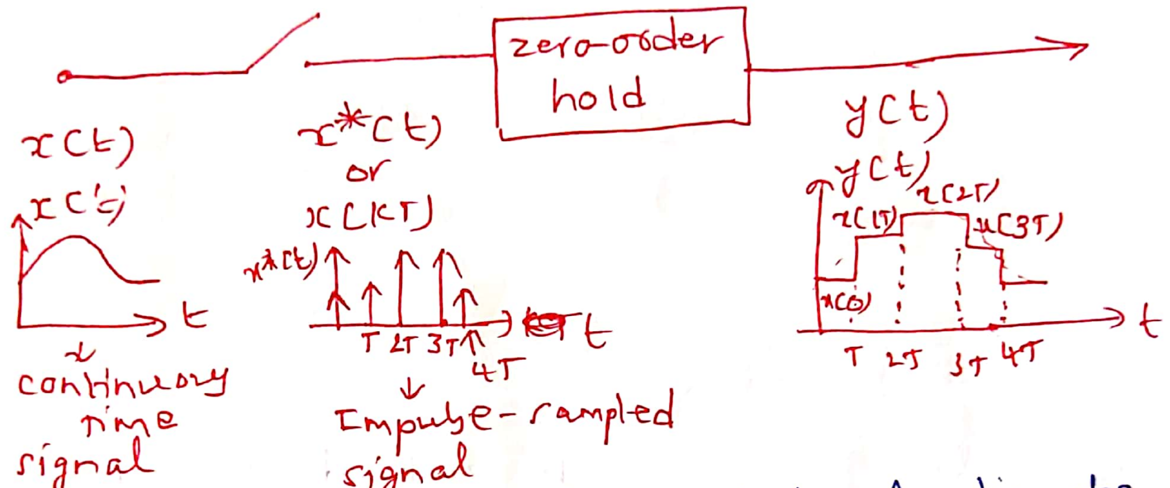
If the data-hold circuit is an n^{th}-order polynomial extrapolator, it is called an n^{th} order hold.

If $n=1$, it is called a First-order hold.

The simplest data-hold is obtained when $n=0$ and it is called zero-order hold circuit.

In this $h(kT+t) = x(kT)$

zero-order hold:



The Input signal is sampled at discrete instants and sampled signal is passed through the zero order hold. The zero-order circuit smoothes the sampled signal to produce the signal $y(t)$.

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)$$

$$L\{x^*(t)\} = x^*(s) = \sum_{k=0}^{\infty} x(kT)\,e^{-kTs}$$

$(\because z = e^{Ts})$

$$x^*(s) = \sum_{k=0}^{\infty} x(kT)\,z^{-k} = X(z) \longrightarrow (1)$$

The output of the ZOH is Rectangular pulses, so $y(t)$ can be written as.

[$\because$ and each pulse can be written ~~as~~ by using rectangular or unit step functions]

$$y(t) = x(0)[u(t) - u(t-T)] + x(1T)[u(t-T) - u(t-2T)] + \cdots + x(kT)[u(t-kT) - u(t-(k+1)T)]$$

$$y(t) = \sum_{k=0}^{\infty} x(kT)\left[u(t-kT) - u(t-(k+1)T)\right]$$

Apply Laplace Transform.

$$Y(s) = \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right]$$

$$Y(s) = \sum_{k=0}^{\infty} x(kT)\, e^{-kTs}\left[\frac{1}{s} - \frac{e^{-Ts}}{s}\right]$$

$$Y(s) = \sum_{k=0}^{\infty} x(kT)\,\frac{e^{-kTs}}{s}\,(1-e^{-Ts})$$

From ①:

$$Y(s) = \cancel{\sum_{k=0}^{\infty}} \frac{x^*(s)}{s}\,(1-e^{-Ts})$$

$$\left(\because x^*(s) = \sum_{k=0}^{\infty} x(kT)\, e^{-kTs}\right)$$

$$Y(s) = \frac{x^*(s)(1-e^{-Ts})}{s}$$

$\frac{Y(s)}{x^*(s)}$ = Transfer function of ZOH $= \frac{1-e^{-Ts}}{s}$

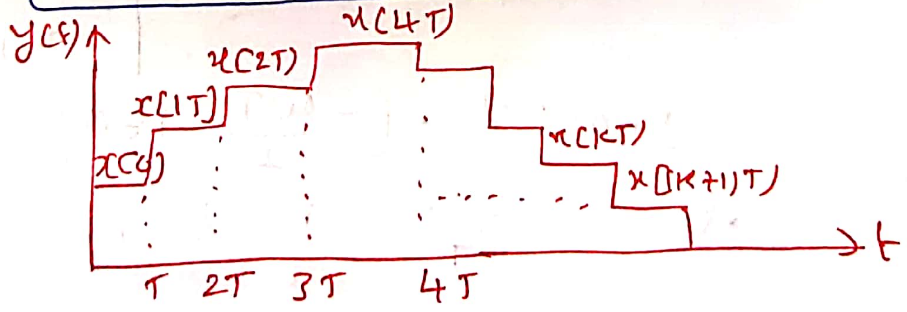
$$\boxed{\frac{Y(s)}{x^*(s)} = \frac{1-e^{-Ts}}{s}}$$

$$\boxed{G_{ho}(s) = \frac{1-e^{-Ts}}{s}}$$

$\left[\because L\{u(t)\} = \frac{1}{s}\right.$

$\left. L\{u(t-kT)\} = \frac{e^{-kTs}}{s}\right]$

Note:



Frequency domain characteristics:

$$G_{ho}(s) = \frac{1-e^{-Ts}}{s}$$

$$G_{ho}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}$$

$$G_{ho}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}$$

$$G_{ho}(j\omega) = \frac{(e^{-j\omega T/2} \cdot e^{j\omega T/2} - e^{-j\omega T})}{j\omega}$$

$$G_{ho}(j\omega) = \frac{e^{-j\omega T/2}(e^{j\omega T/2} - e^{-j\omega T/2})}{j\omega}$$

$$G_{ho}(j\omega) = \frac{e^{-j\omega T/2}}{\omega}\left(\frac{e^{j(\omega T/2)} - e^{-j(\frac{\omega T}{2})}}{j}\right)$$

$$G_{ho}(j\omega) = \frac{(e^{-j\omega T/2})2}{\omega}\left(\frac{e^{j(\omega T/2)} - e^{-j(\omega T/2)}}{2j}\right)$$

$$\left(\because \sin x = \frac{e^{jx} - e^{-jx}}{2j}\right)$$

$$G_{ho}(j\omega) = \frac{2e^{-j\omega T/2}}{\omega}\left(\sin\left(\frac{\omega T}{2}\right)\right)$$

$$G_{ho}(j\omega) = \frac{Te^{-j\omega T/2}}{\left(\frac{T\omega}{2}\right)} \sin(\omega T/2)$$

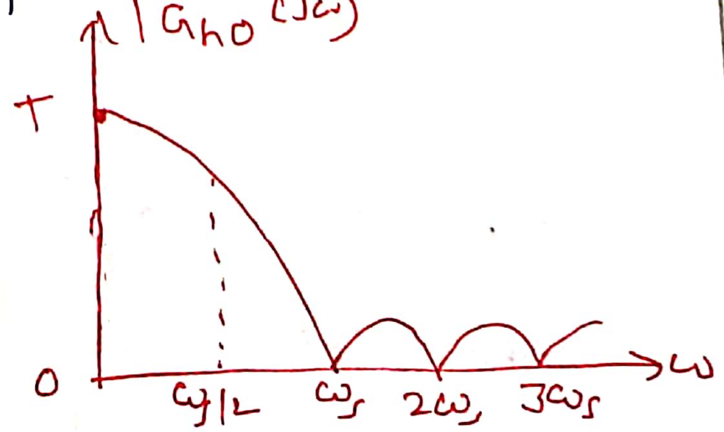
$$G_{ho}(j\omega) = (T\,e^{-j\omega T/2})\frac{\sin(\omega T/2)}{(\omega T/2)}$$

Magnitude: $|G_{ho}(j\omega)| = T\left|\frac{\sin(\omega T/2)}{\omega T/2}\right|$

phase: $\angle G_{ho}(j\omega) = \left(-\frac{\omega T}{2} - 180\right)$

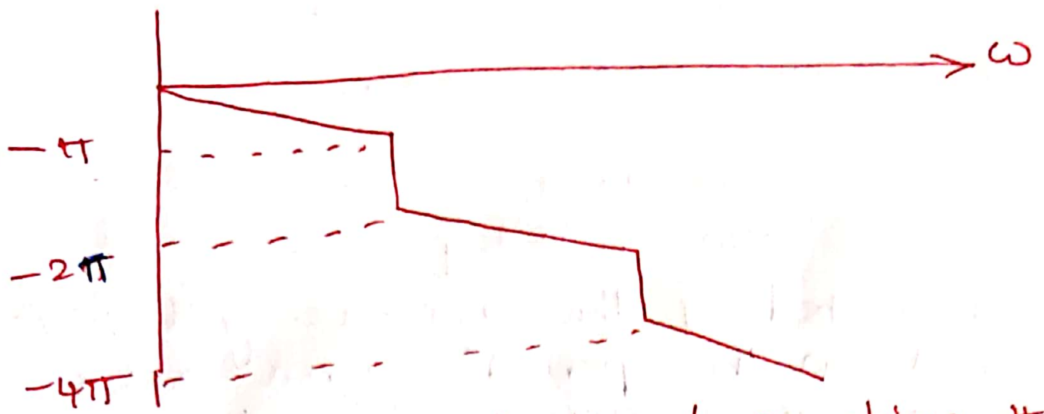
at $\omega = \frac{2\pi k}{T}$, $k = 1, 2, \cdots$

Magnitude plot.



Scanned by CamScanner

phase plot:



Background to understand sampling theorem:

* Exponential Fourier series of periodic signals:

Fourier → Trigonometric form, cosine form, exponential form.

Here we use exponential form.

$x(t)$ is a periodic signal

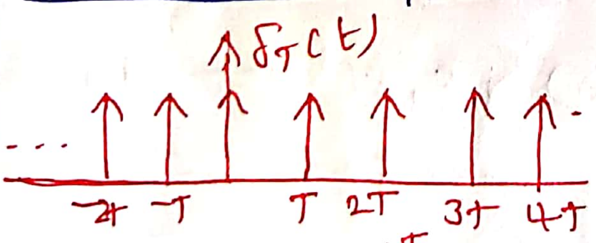
$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

C_n = Fourier series co-efficient

$$\boxed{C_n = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jn\omega_0 t}\, dt}$$

$$\omega_0 = \frac{2\pi}{T}$$

Fourier series of Impulse train:



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T \delta_T(t)\, e^{-jn\omega_0 t}\, dt$$

$$\left(\because \int_0^T \delta_T(t) e^{-jn\omega_0 t} dt = 1\right)$$

$$\boxed{C_n = \frac{1}{T}}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_0 t}$$

Fourier Transform:

Fourier series → For periodic signal

Fourier Transform → Aperiodic signal, Bandlimited periodic signal

~~Bandwidth~~

Fourier Transform of bandlimited signal:

$$F.T = 2\pi c_n \times \sum_{n=-\alpha}^{\alpha} \delta(\omega - n\omega_0)$$

c_n = Fourier series co-efficient of $x(t)$

$x(t)$ = band limited periodic signal.

Band limited is the frequency highest component in the periodic signal after writing Fourier-transform. It must be a finite number

Eg: fourier transform of $\cos\omega_0 t$

$\cos\omega_0 t \rightarrow$ periodic & bandlimited signal.

exponential Fourier series of $x(t) = \cos\omega_0 t$

$$x(t) = \cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$x(t) = 0.5e^{j\omega_0 t} + 0.5e^{-j\omega_0 t} \rightarrow (1)$$

$$x(t) = \sum_{n=-\alpha}^{\alpha} c_n e^{j\omega_0 t n} = \sum_{n=-\alpha}^{\alpha} c_n e^{jn\omega_0 t}$$

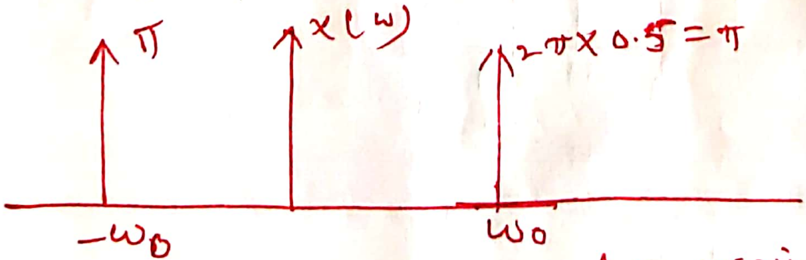
$$= \ldots\ldots c_{-1}e^{-j\omega_0 t} + c_0 + c_1 e^{j\omega_0 t} + \ldots \rightarrow (2)$$

compare (1) and (2):

$c_1 = 0.5$ $\quad$ $c_0 = 0$ $\quad$ $(n = 1, -1)$

$$FT = 2\pi \sum_{n=-\alpha}^{\alpha} c_n \delta(\omega - n\omega_0)$$

$$FT(x(t)) = 2\pi\left[c_{-1}\delta(\omega + \omega_0) + c_1\delta(\omega - \omega_0)\right]$$



Fourier-Transform of Impulse Train:

$$F.T[\delta_T(t)] = 2\pi \sum_{n=-\alpha}^{\alpha} c_n \delta(\omega - n\omega_0)$$

$$\delta_\omega(\omega) = 2\pi \sum_{n=-\alpha}^{\alpha} \frac{1}{T}\delta(\omega - n\omega_0)$$

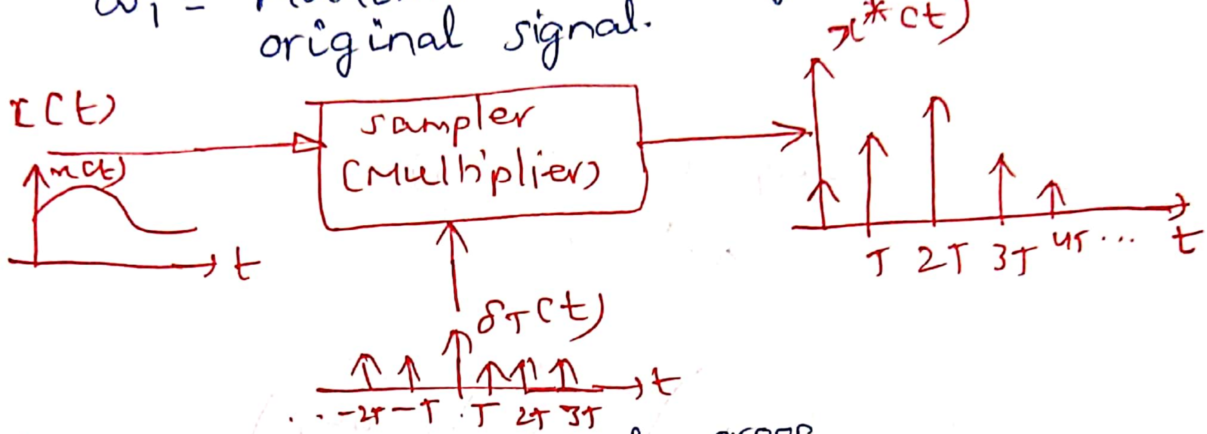
Sampling theorem: (Shanon's sampling theorem)

Statement: The Reconstruction of Band-limited sampled signal is possible if the sampling frequency is atleast double to the maximum frequency component of original signal"

i,e $\boxed{\omega_s \geq 2\omega_1}$

ω_s = sampling frequency.

ω_1 = Maximum frequency component in original signal.



Proof: It is a graphical proof.

$$x^*(t) = x(t)\, \delta_T(t)$$

Apply F.T

$$F.T[x^*(t)] = F.T[x(t)\, \delta_T(t)] \rightarrow ①$$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$\delta_T(t) \xleftrightarrow{FT} \delta_{\omega}(\omega)$$

$$x(t)\delta_T(t) \xleftrightarrow{F.T} \frac{1}{2\pi}\left[X(\omega) * \delta_{\omega_s}(\omega)\right]$$

↙ convolution symbol

From ①:

$$X^*(\omega) = \frac{1}{2\pi}\, X(\omega) * \delta\omega_s(\omega)$$

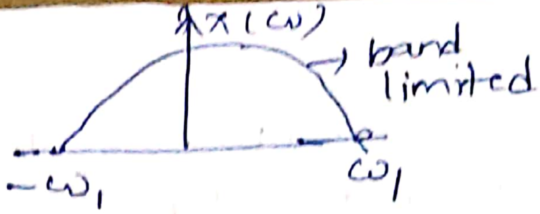
$$= \frac{1}{2\pi}\left[X(\omega) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-\omega_s n)\right]$$

$$\left(\because\ \delta\omega_s(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-\omega_s n)\right]$$

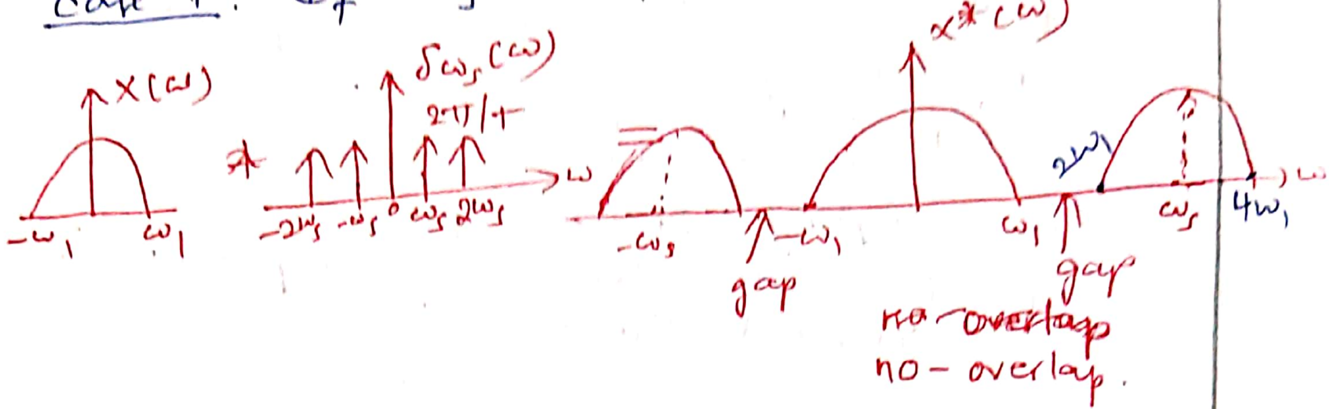
$$\boxed{X^*(\omega) = \frac{1}{2\pi}\left[X(\omega) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta[\omega-\omega_s n]\right]}$$

$\rightarrow ②$

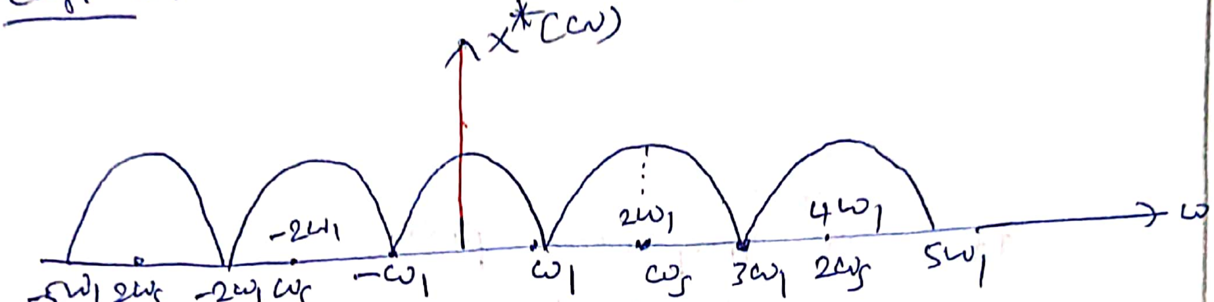
$X(\omega) = F.T[x(t)] =$



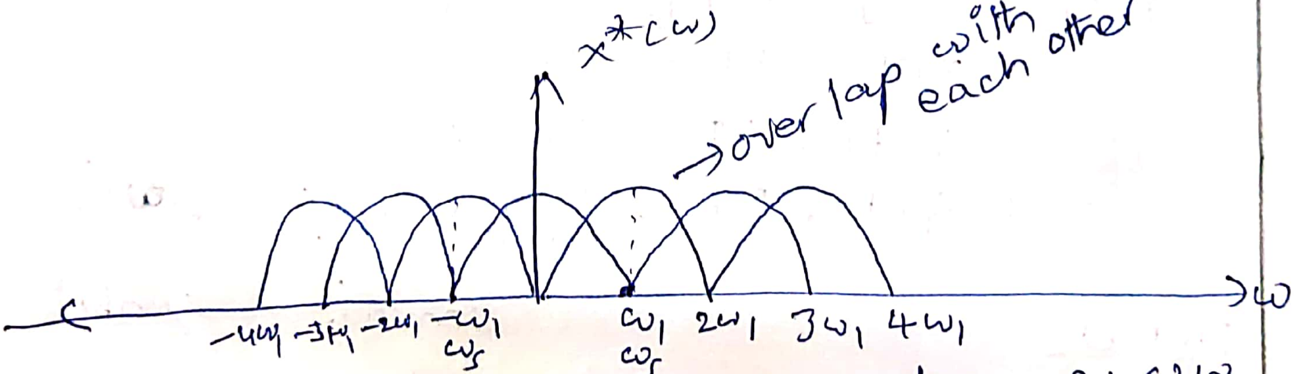
<u>case-1</u>: If $\omega_s = 3\omega_1$ $(\omega_s > 2\omega_1)$



<u>case-2</u>: $\omega_s = 2\omega_1$ $(\omega_s \geq 2\omega_1)$



<u>case-3</u>: $\omega_s = \omega_1$ $(\omega_s < 2\omega_1)$



so that from the graphs when $\omega_s < 2\omega_1$ there is a ~~aliasing~~ overlapping, it is called aliasing. and also it is clear that ~~there~~ "the signal reconstruction is possible when the sampling frequency is greater than the two times the maximum frequency component of the signal."