ELECTROMAGNETIC FIELDS AND WAVES

LECTURE NOTES

B.TECH
(II YEAR – II SEM)
(2020-21)

Prepared by:

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

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(Affiliated to JNTUH, Hyderabad, Approved by AICTE - Accredited by NBA & NAAC – 'A' Grade - ISO 9001:2015 Certified)

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MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

II Year B.Tech. ECE-II Sem	**L**	**T/P/D C**
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(R18A0406) ELECTROMAGNETIC FIELDS & WAVES

OBJECTIVES

The course objectives are:

1. To introduce the student to the coordinate system and its implementation to electromagnetics.
2. To elaborate the concept of electromagnetic waves and their practical applications.
3. To study the propagation, reflection, and refraction of plane waves in different media.

UNIT - I:

Vector Analysis & Co-ordinate system: Vector analysis- Representation, operations-Dot product and cross product, Basics of coordinate system- rectangular, cylindrical and spherical co-ordinate systems.

Electrostatics-I: Coulomb's Law, Electric Field Intensity - Fields due to Different Charge Distributions, Electric Flux Density; Illustrative Problems.

UNIT - II:

Electrostatics-II:

Gauss Law and Applications, Electric Potential, Relations Between E and V, Maxwell's Equations for Electrostatic Fields, Dielectric Constant, Isotropic and Homogeneous Dielectrics, Continuity Equation, Relaxation Time, Poisson's and Laplace's Equations, Boundary conditions-conductor-Dielectric and Dielectric-Dielectric; Illustrative Problems.

UNIT - III:

Magnetostatics: Biot - Savart's Law , Ampere's Circuital Law and Applications, Magnetic Flux Density, Maxwell's Equations for Magnetostatic Fields, Magnetic Scalar and Vector Potentials, Ampere's Force law **,** Faraday's Law, Displacement Current Density, Maxwell's Equations for time varying fields, Illustrative Problems.

UNIT - IV:

EM Wave Characteristics-I : Wave Equations for Conducting and Perfect Dielectric Media, Uniform Plane Waves - Definition, Relation Between E & H, Wave Propagation in Lossless and Conducting Media, Wave Propagation in Good Conductors and Good Dielectrics, Illustrative Problems.

UNIT - V:

EM Wave Characteristics – II: Reflection and Refraction of Plane Waves – Normal incidence for both perfect Conductors and perfect Dielectrics, Brewster Angle, Critical Angle and Total Internal Reflection, Surface Impedance, Poynting Vector and Poynting Theorem – Applications, Illustrative Problems.

TEXT BOOKS:

1. Elements of Electromagnetics - Matthew N. O. Sadiku, 4th., Oxford Univ. Press.
2. Electromagnetic Waves and Radiating Systems - E.C. Jordan and K. G. Balmain, 2nd Ed., 2000, PHI.
3. Engineering Electromagnetic - William H. Hay Jr. and John A. Buck, 7thEd., 2006, TMH

REFERENCES BOOKS:

1. Engineering Electromagnetics - Nathan Ida, 2ndEd., 2005, Springer (India) Pvt. Ltd., New Delhi.
2. Electromagnetic Waves and Transmission Lines-Y Mallikarjuna Reddy, University Press.
3. Electromagnetic Fields Theory and Transmission Lines - G. Dashibhushana Rao, Wiley India, 2013.

COURSE OUTCOMES:

Upon the successful completion of the course, students will be able to;

1. Study time varying Maxwell equations and their applications in electromagnetic problems
2. Determine the relationship between time varying electric and magnetic field and electromotive force
3. Use Maxwell equation to describe the propagation of electromagnetic waves
4. Demonstrate the reflection and refraction of waves at boundaries

UNIT – I
ELECTROSTATICS

Contents

Vector Analysis & Co-ordinate system

- Vector analysis
  - Representation
  - Operations-Dot product and cross product
- Basics of coordinate system
  - Rectangular coordinate system
  - Cylindrical coordinate system
  - Spherical coordinate system

Electrostatics-I:

- Coulomb’s Law
- Electric Field Intensity
  - Fields due to Different Charge Distributions
- Electric Flux Density
- Problems.

Vector Analysis

Introduction:

Vector Algebra is a part of algebra that deals with the theory of vectors and vector spaces.

Most of the physical quantities are either scalar or vector quantities.

Scalar Quantity:

Scalar is a number that defines magnitude. Hence a scalar quantity is defined as a quantity that has magnitude only. A scalar quantity does not point to any direction i.e. a scalar quantity has no directional component.

For example when we say, the temperature of the room is 30o C, we don't specify the direction.

Hence examples of scalar quantities are mass, temperature, volume, speed etc.

A scalar quantity is represented simply by a letter – A, B, T, V, S.

Vector Quantity:

A Vector has both a magnitude and a direction. Hence a vector quantity is a quantity that has both magnitude and direction.

Examples of vector quantities are force, displacement, velocity, etc.

$\vec{\mathbf{A}}, \vec{\mathbf{V}}, \vec{\mathbf{B}}, \vec{\mathbf{F}}$

A vector quantity is represented by a letter with an arrow over it or a bold letter.

Unit Vectors:

When a simple vector is divided by its own magnitude, a new vector is created known as the unit vector. A unit vector has a magnitude of one. Hence the name - unit vector.

A unit vector is always used to describe the direction of respective vector.

$$\mathbf{a}_A = \frac{\vec{A}}{|A|} \Rightarrow \vec{A} = |A|\,\mathbf{a}_A$$

Hence any vector can be written as the product of its magnitude and its unit vector. Unit Vectors along the co-ordinate directions are referred to as the base vectors. For example unit vectors along X, Y and Z directions are ax, ay and az respectively.

Position Vector / Radius Vector ($\overline{OP}$):

A Position Vector / Radius vector define the position of a point(P) in space relative to the origin(O).Hence Position vector is another way to denote a point in space.

$$\overline{OP} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$$

Displacement Vector

Displacement Vector is the displacement or the shortest distance from one point to another.

Vector Multiplication

When two vectors are multiplied the result is either a scalar or a vector depending on how they are multiplied. The two important types of vector multiplication are:

- Dot Product/Scalar Product (A.B)
- Cross product (A x B)

1. DOT PRODUCT (A. B):

Dot product of two vectors A and B is defined as:

$$\bar{A}.\bar{B} = |\bar{A}|\,|\bar{B}| \cos\theta_{AB}$$

Where θ_{AB} is the angle formed between A and B.
Also θ_{AB} ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$
The result of A.B is a scalar, hence dot product is also known as Scalar Product.

Properties of Dot Product:

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then

$$\bar{A}.\bar{B} = AxBx + AyBy + AzBz$$

2. $\bar{A}.\bar{B} = |A|\,|B|$, if $\cos\theta_{AB}=1$ which means $\theta_{AB} = 0^0$

This shows that A and B are in the same direction or we can also say that A and B are parallel to each other.

3. $\bar{A}.\bar{B} = -|A|\,|B|$, if $\cos\theta_{AB}=-1$ which means $\theta_{AB} = 180^0$.
This shows that A and B are in the opposite direction or we can also say that A and B are antiparallel to each other.

4. $\bar{A}.\bar{B} = 0$, if $\cos\theta_{AB}=0$ which means $\theta_{AB} = 90^0$.
This shows that A and B are orthogonal or perpendicular to each other.

5. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x.\bar{a}_x = \bar{a}_y.\bar{a}_y = \bar{a}_z.\bar{a}_z = 1$$

$$\bar{a}_x.\bar{a}_y = \bar{a}_y.\bar{a}_z = \bar{a}_z.\bar{a}_x = 0$$

2. Cross Product (A X B):

Cross Product of two vectors A and B is given as:

$$\bar{A} X \bar{B} = \left|\bar{A}\right| \left|\bar{B}\right| \sin\theta_{AB}\, \bar{a}_n$$

Where θ_{AB} is the angle formed between A and B and $\bar{a}_n$ is a unit vector normal to both A and B. Also θ ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$

The cross product is an operation between two vectors and the output is also a vector.

Properties of Cross Product:

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then,

$$\mathbf{A} * \mathbf{B} = \begin{vmatrix} \mathbf{a_x} & \mathbf{a_y} & \mathbf{a_z} \\ \mathbf{A_x} & \mathbf{A_y} & \mathbf{A_z} \\ \mathbf{B_x} & \mathbf{B_y} & \mathbf{B_z} \end{vmatrix}$$

The resultant vector is always normal to both the vector A and B.

2. $\bar{A} X \bar{B} = 0$, if $\sin\theta_{AB} = 0$ which means $\theta_{AB} = 0^0$ or 180^0;
This shows that A and B are either parallel or antiparallel to each other.

3. $\bar{A} X \bar{B} = \left|\bar{A}\right| \left|\bar{B}\right| \bar{a}_n$, if $\sin\theta_{AB} = 0$ which means $\theta_{AB} = 90^0$.
This shows that A and B are orthogonal or perpendicular to each other.

4. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x X \bar{a}_x = \bar{a}_y X \bar{a}_y = \bar{a}_z X \bar{a}_z = 0$$

$$\bar{a}_x X \bar{a}_y = \bar{a}_z \ , \bar{a}_y X \bar{a}_z = \bar{a}_x \ , \ \bar{a}_z X \bar{a}_x = \bar{a}_y$$

CO-ORDINATE SYSTEMS:

Co-Ordinate system is a system of representing points in a space of given dimensions by coordinates, such as the Cartesian coordinate system or the system of celestial longitude and latitude.

In order to describe the spatial variations of the quantities, appropriate coordinate system is required. A point or vector can be represented in a curvilinear coordinate system that may be orthogonal or non-orthogonal. An orthogonal system is one in which the coordinates are mutually perpendicular to each other.

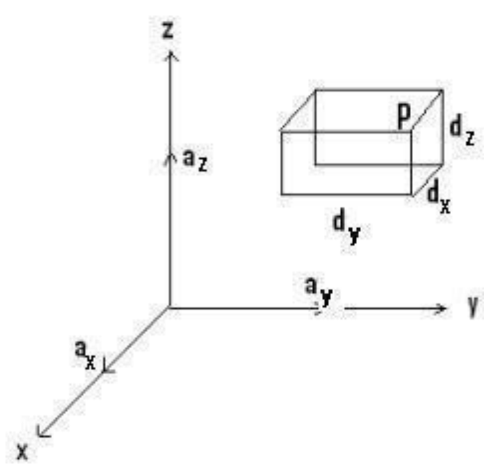
The different co-ordinate system available are:

- Cartesian or Rectangular co-ordinate system.(Example: Cube, Cuboid)
- Circular Cylindrical co-ordinate system.(Example : Cylinder)
- Spherical co-ordinate system. (Example: Sphere)

The choice depends on the geometry of the application.

A set of 3 scalar values that define position and a set of unit vectors that define direction form a co-ordinate system. The 3 scalar values used to define position are called co-ordinates. All coordinates are defined with respect to an arbitrary point called the origin.

1. Cartesian Co-ordinate System / Rectangular Co-ordinate System (x,y,z)



A Vector in Cartesian system is represented as (Ax, Ay, Az) Or

$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$$

Where $\bar{a}_x, \bar{a}_y$ and $\bar{a}_z$ are the unit vectors in x, y, z direction respectively.

Range of the variables:

It defines the minimum and the maximum value that x, y and z can have in Cartesian system.
$-\infty \leq$ x,y,z $\leq \infty$

Differential Displacement / Differential Length (dl):

It is given as

$\bar{dl} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$

Differential length for a line parallel to x, y and z axis are respectively given as:

dl = $dx\bar{a}_x$---(For a line parallel to x-axis).
dl = $dy\bar{a}_y$ ---(For a line Parallel to y-axis).
dl = $dz\bar{a}_z$ ---(For a line parallel to z-axis).

If there is a wire of length L in z-axis, then the differential length is given as dl = dz az. Similarly if the wire is in y-axis then the differential length is given as dl = dy ay.

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface.
The differential surface (area element) is defined as

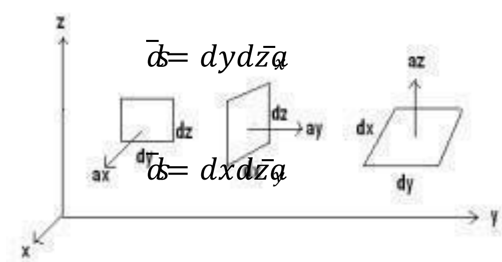
$$\bar{ds} = ds\bar{a}_n$$

Where $\bar{a}_n$, is the unit vector perpendicular to the surface.

For the 1st figure, $\bar{ds} = dydz\bar{a}_x$

2nd figure, $\bar{ds} = dxdz\bar{a}_y$

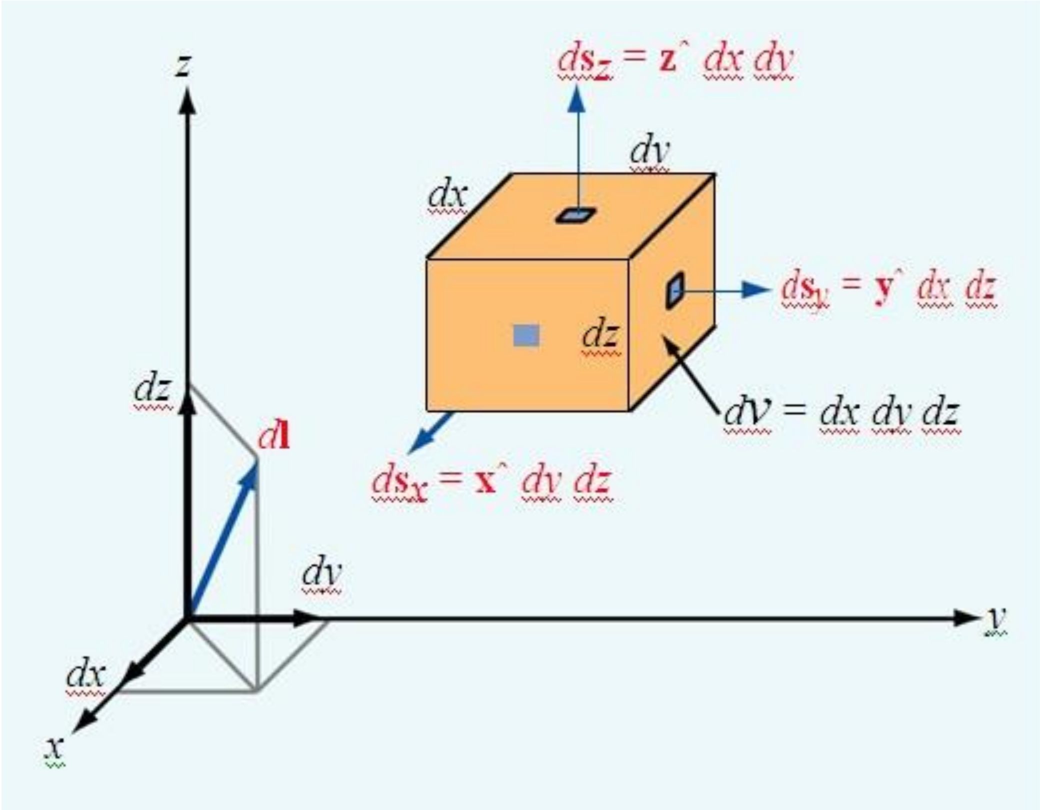
3rd figure, $\bar{ds} = dxdy\bar{a}_z$



Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = dxdydz$$



Differential length, area, and volume in Cartesian coordinates.

2. Circular Cylindrical Co-ordinate System

A Vector in Cylindrical system is represented as (A_r, A_ϕ, A_z) or

$$\bar{A} = A_r\bar{a}_r + A_\phi\bar{a}_\phi + A_z\bar{a}_z$$

Where $\bar{a}_r$, $\bar{a}_\phi$ and $\bar{a}_z$ are the unit vectors in r, Φ and z directions respectively.

The physical significance of each parameter of cylindrical coordinates:

1. The value r indicates the distance of the point from the z-axis. It is the radius of the cylinder.
2. The value Φ, also called the azimuthal angle, indicates the rotation angle around the z-axis. It is basically measured from the x axis in the x-y plane. It is measured anti clockwise.
3. The value z indicates the distance of the point from z-axis. It is the same as in the Cartesian system. In short, it is the height of the cylinder.

Range of the variables:

It defines the minimum and the maximum values of r, Φ and z.

$0 \leq r \leq \infty$
$0 \leq \Phi \leq 2\pi$
$-\infty \leq z \leq \infty$

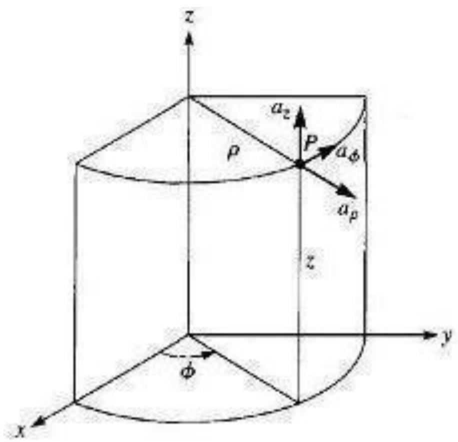


Figure shows Point P and Unit vectors in Cylindrical Co-ordinate System.

Differential Displacement / Differential Length (dl):

It is given as

$$\bar{dl} = dr\bar{a}_r + rd\varphi\bar{a}_\phi + dz\bar{a}_z$$

Differential length for a line parallel to r, Φ and z axis are respectively given as:

dl = $dr\bar{a}_r$---(For a line parallel to r-direction).
dl = $rd\varphi\bar{a}_\phi$ ---(For a line Parallel to Φ-direction).
dl = $dz\bar{a}_z$ ---(For a line parallel to z-axis).

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface. The differential surface (area element) is defined as

$$\bar{ds} = ds\bar{a}_N$$

Where $\bar{a}_N$, is the unit vector perpendicular to the surface.

This surface describes a circular disc. Always remember- To define a circular disk we need two parameter one distance measure and one angular measure. An angular parameter will always give a curved line or an arc.

In this case dΦ is measured in terms of change in arc.

Arc is given as:
Arc= radius * angle

$$\bar{ds} = rdrd\varphi\bar{a}_z$$
$$\bar{ds} = drdz\bar{a}_\varphi$$
$$\bar{ds} = rdrd\varphi\bar{a}_z$$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = rdrd\varphi dz$$

3. Spherical coordinate System:

Spherical coordinates consist of one scalar value (r), with units of distance, while the other two scalarvalues (θ, Φ) have angular units (degrees or radians).

A Vector in Spherical System is represented as (A_r, A_Θ, A_Φ) or

$$\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\varphi\bar{a}_\varphi$$

Where $\bar{a}_r, \bar{a}_\theta$ and $\bar{a}_\varphi$ are the unit vectors in r, θ and Φ direction respectively.

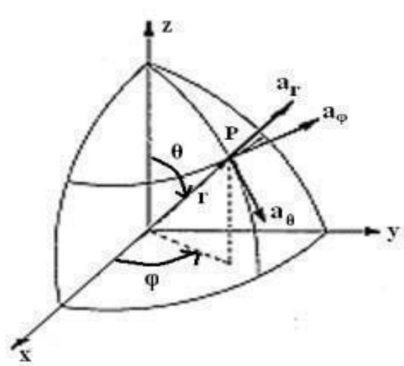
The physical significance of each parameter of spherical coordinates:

1. The value r expresses the distance of the point from origin (i.e. similar to altitude). It is the radius of the sphere.
2. The angle θ is the angle formed with the z- axis (i.e. similar to latitude). It is also called the co-latitude angle. It is measured clockwise.
3. The angle Φ, also called the azimuthal angle, indicates the rotation angle around the z-axis (i.e. similar to longitude). It is basically measured from the x axis in the x-y plane. It is measured counter-clockwise.

Range of the variables:

It defines the minimum and the maximum value that r, θ and υ can have in spherical co-ordinate system.

$0 \leq r \leq \infty$
$0 \leq \theta \leq \pi$
$0 \leq \Phi \leq 2\pi$



Differential length:

It is given as

$\bar{dl} = dr\bar{a}_r + rd\theta\bar{a}_\theta + r\sin\theta\, d\varphi\bar{a}_\phi$

Differential length for a line parallel to r, θ and Φ axis are respectively given as:

dl = $dr\bar{a}_r$--(For a line parallel to r axis)

dl = $rd\theta\bar{a}_\theta$---(For a line parallel to θ direction)

dl = $r\sin\theta\, d\varphi\bar{a}_\phi$ --(For a line parallel to Φ direction)

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface. The differential surface (area element) is defined as

$$\bar{ds} = ds\bar{a}_n$$

Where $\bar{a}_n$, is the unit vector perpendicular to the surface.

$$\bar{ds} = rdrd\theta\bar{a}_\phi$$
$$\bar{ds} = r^2\sin\theta\, d\varphi d\theta\bar{a}_r$$
$$\bar{ds} = r\sin\theta\, drd\varphi\bar{a}_\theta$$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = r^2\sin\theta\, drd\varphi d\theta$$

Coordinate transformations:

Coordinate transformations

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Vector relations in the three common coordinate systems.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of $\mathbf{A}$ $\quad \lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\quad \overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\quad \mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\quad \mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $\quad d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $\quad d\mathcal{V} =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Del operator:

Del is a vector differential operator. The del operator will be used in for differential operations throughout any course on field theory. The following equation is the del operator for different coordinate systems.

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z = \nabla_{x,y,z}$$

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

Gradient of a Scalar:

• The gradient of a scalar field, V, is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z = \nabla V_{x,y,z}$$

• To help visualize this concept, take for example a topographical map. Lines on the map represent equal magnitudes of the scalar field. The gradient vector crosses map at the location where the lines packed into the most dense space and perpendicular (or normal) to them. The orientation (up or down) of the gradient vector is such that the field is increased in magnitude along that direction.

-Fundamental properties of the gradient of a scalar field

– The magnitude of gradient equals the maximum rate of change in V per unit distance

– Gradient points in the direction of the maximum rate of change in V

– Gradient at any point is perpendicular to the constant V surface that passes through that point

– The projection of the gradient in the direction of the unit vector **a**, is

$$\nabla V \cdot \hat{a}$$

and is called the directional derivative of V along **a**. This is the rate of change of V in the direction of **a**.

– If **A** is the gradient of V, then V is said to be the scalar potential of **A.**

Divergence of a Vector:

• The divergence of a vector, **A**, at any given point P is the outward flux per unit volume as volume shrinks about P.

$$div\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot d\vec{S}}{\Delta v}$$

Divergence Theorem:

• The divergence theorem states that the total outward flux of a vector field, **A**, through the closed surface, S, is the same as the volume integral of the divergence of **A**.

• This theorem is easily shown from the equation for the divergence of a vector field.

$$\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$$

$$div\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot d\vec{S}}{\Delta v}$$

$$\int_v \nabla \cdot \vec{A} dv = \oint_s \vec{A} \cdot d\vec{S}$$

Curl of a Vector:

The curl of a vector, **A** is an axial vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

-Curl of a vector in each of the three primary coordinate systems are,

Cartesian
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{a}_z$$

Cylindrical
$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{a}_\rho - \left[\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z}\right]\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\hat{a}_z$$

Spherical
$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\hat{a}_r - \frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi}\right]\hat{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{a}_\phi$$

Stokes Theorem:

• Stokes theorem states that the circulation of a vector field A, around a closed path, L is equal to the surface integral of the curl of A over the open surface S bounded by L. This theorem has been proven to hold as long as **A** and the curl of **A** are continuous along the closed surface S of a closed path L

• This theorem is easily shown from the equation for the curl of a vector field.

$$\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$$

$$curl\vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{l}}{\Delta S} \right)_{\max} \hat{a}_n$$

$$\oint_L \vec{A} \cdot d\vec{l} - \oint_s \left(\nabla \times \vec{A}\right) \cdot d\vec{S}$$

__Classification of vector field:__

The vector field, **A**, is said to be divergenceless (or solenoidal) if $\nabla \cdot \vec{A} = 0$.

– Such fields have no source or sink of flux, thus all the vector field lines entering an enclosed surface, S, must also leave it.

– Examples include magnetic fields, conduction current density under steady state, and imcompressible fluids

– The following equations are commonly utilized to solve divergenceless field problems

$$\nabla \cdot \vec{A} = 0$$

$$\oint_S \vec{A} \cdot d\vec{S} = \int_v \left(\nabla \cdot \vec{A}\right) dv = 0$$

$$\vec{F} = \nabla \times \vec{A}$$

The vector field, **A**, is said to be potential (or irrotational) if $\nabla \times \vec{A} = 0$

– Such fields are said to be conservative. Examples include gravity, and electrostatic fields.

– The following equations are commonly used to solve potential field problems;

$$\nabla \times \nabla V = 0 \qquad \oint_L \vec{A} \cdot d\vec{l} = \oint_S \left(\nabla \times \vec{A}\right) \cdot d\vec{S} = 0$$

$$\nabla \times \vec{A} = 0 \qquad \vec{A} = -\nabla V$$

Electrostatics-I

Introduction:

Electromagnetic theory is concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering. Electromagnetic theory is also required for the understanding, analysis and design of various electrical, electromechanical and electronic systems.

Electromagnetic theory can be thought of as generalization of circuit theory. Electromagnetic theory deals directly with the electric and magnetic field vectors where as circuit theory deals with the voltages and currents. Voltages and currents are integrated effects of electric and magnetic fields respectively.

Electromagnetic field problems involve three space variables along with the time variable and hence the solution tends to become correspondingly complex. Vector analysis is the required mathematical tool with which electromagnetic concepts can be conveniently expressed and best comprehended. Since use of vector analysis in the study of electromagnetic field theory is prerequisite, first we will go through vector algebra.

Applications of Electromagnetic theory:

This subject basically consist of static electric fields, static magnetic fields, time-varying fields & it' applications. One of the most common applications of electrostatic fields is the deflection of a charged particle such as an electron or proton in order to control it's trajectory. The deflection is achieved by maintaining a potential difference between a pair of parallel plates. This principle is used in CROs, ink-jet printer etc. Electrostatic fields are also used for sorting of minerals for example in ore separation. Other applications are in electrostatic generator and electrostatic voltmeter.

The most common applications of static magnetic fields are in dc machines. Other applications include magnetic deflection, magnetic separator, cyclotron, hall effect sensors, magneto hydrodynamic generator etc.

Electrostatics is a branch of science that involves the study of various phenomena caused by electric charges that are slow-moving or even stationary. Electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics as the study of electric charges at rest.

The two important laws of electrostatics are

- Coulomb‘s Law.
- Gauss‘s Law.

Both these laws are used to find the electric field due to different charge configurations.

Coulomb's law is applicable in finding electric field due to any charge configurations where as Gauss's law is applicable only when the charge distribution is symmetrical.

Coulomb's Law

Coulomb's Law states that the force between two point charges Q1and Q2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

A point charge is a charge that occupies a region of space which is negligibly small compared to the distance between the point charge and any other object.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

Mathematically, $F = \frac{kQ_1Q_2}{R^2}$, where k is the proportionality constant.

In SI units, Q1 and Q2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$, ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1Q_2}{R^2}$ ....(1)

As shown in the Figure 1 let the position vectors of the point charges Q1and Q2 are given by $\vec{r}_1$ and $\vec{r}_2$. Let $\overrightarrow{F_{12}}$ represent the force on Q1 due to charge Q2.

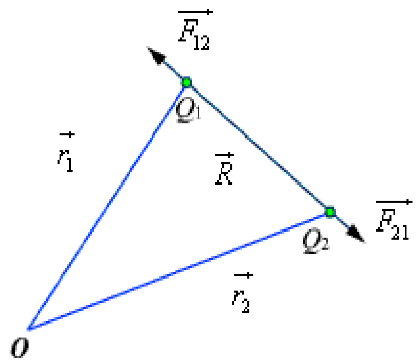


Fig 1: Coulomb's Law

The charges are separated by a distance of $R = |\vec{r}_1 - \vec{r}_2| = |\vec{r}_2 - \vec{r}_1|$. We define the unit vectors as

$\hat{a}_{12} = \frac{(\vec{r}_2 - \vec{r}_1)}{R}$ and $\hat{a}_{21} = \frac{(\vec{r}_1 - \vec{r}_2)}{R}$

$\vec{F}_{12}$ can be defined as $\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$.

Similarly the force on Q_1 due to charge Q_2 can be calculated and if $\vec{F}_{21}$ represents this force then we can write $\vec{F}_{21} = -\vec{F}_{12}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have N number of charges $Q_1, Q_2, \ldots\ldots\ldots Q_N$ located respectively at the points represented by the position vectors $\vec{r}_1, \vec{r}_2, \ldots\ldots \vec{r}_N$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

<u>Field:</u>

A field is a function that specifies a particular physical quantity everywhere in a region. Depending upon the nature of the quantity under consideration, the field may be a vector or a scalar field. Example of scalar field is the electrostatic potential in a region while electric or magnetic fields at any point is the example of vector field.

Static Electric Fields:

Electrostatics can be defined as the study of electric charges at rest. Electric fields have their sources in electric charges. The fundamental & experimentally proved laws of electrostatics are Coulomb's law & Gauss's theorem.

Electric Field:

Electric field due to a charge is the space around the unit charge in which it experiences a force. Electric field intensity or the electric field strength at a point is defined as the force per unit charge.

Mathematically,

E = F / Q

OR

F = E Q

The force on charge Q is the product of a charge (which is a scalar) and the value of the electric field (which is a vector) at the point where the charge is located. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q}$$

The electric field intensity *E* at a point *r* (observation point) due a point charge *Q* located at $\vec{r}\,'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3}$$

For a collection of *N* point charges $Q_1, Q_2, \ldots\ldots\ldots Q_N$ located at $\vec{r}_1, \vec{r}_2, \ldots\ldots \vec{r}_N$, the electric field intensity at point $\vec{r}\,'$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3}$$

The expression (6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2 we consider a continuous volume distribution of charge *(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r}\,')dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} = \frac{\rho(\vec{r}\,')dv'(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3}$$

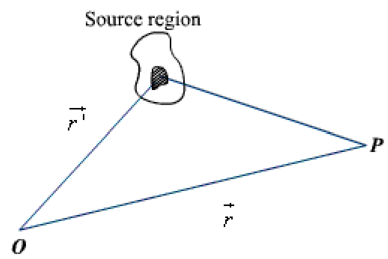


Fig 2: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}\,'\right|^3} dv' \quad \text{...............volume charge..........................}$$

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r}\,')(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}\,'\right|^3} dl' \quad \text{....................line charge ....}$$

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r}\,')(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}\,'\right|^3} ds' \quad \text{.................surface charge.....................}$$

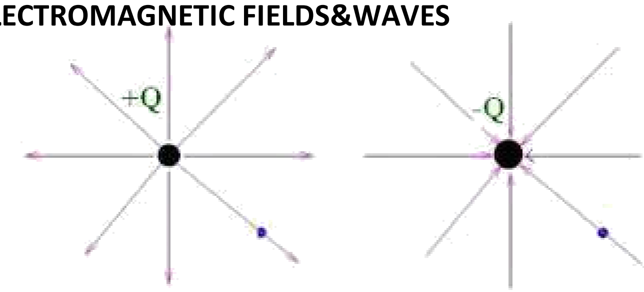
Electric Lines of Forces:

Electric line of force is a pictorial representation of the electric field.

Electric line of force (also called Electric Flux lines or Streamlines) is an imaginary straight or curved path along which a unit positive charge tends to move in an electric field.

Properties Of Electric Lines Of Force:

1. Lines of force start from positive charge and terminate either at negative charge or move to infinity.
2. Similarly lines of force due to a negative charge are assumed to start at infinity and terminate at the negative charge.



3. The number of lines per unit area, through a plane at right angles to the lines, is proportional to the magnitude of E. This means that, where the lines of force are close together, E is large and where they are far apart E is small.
4. If there is no charge in a volume, then each field line which enters it must also leave it.
5. If there is a positive charge in a volume then more field lines leave it than enter it.
6. If there is a negative charge in a volume then more field lines enter it than leave it.
7. Hence we say Positive charges are sources and Negative charges are sinks of the field.
8. These lines are independent on medium.
9. Lines of force never intersect i.e. they do not cross each other.
10. Tangent to a line of force at any point gives the direction of the electric field E at that point.

Electricfluxdensity:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon \vec{E}$$

We define the electric flux as

$$\psi = \int_s \vec{D} . d\vec{s}$$

Solved problems:

Problem1:

Find the charge in the volume defined by $0 \le x \le 1$ m, $0 \le y \le 1$ m, and $0 \le z \le 1$ m if $\rho = 30x^2y$ (μC/m^3). What change occurs for the limits $-1 \le y \le 0$ m?

Since $dQ = \rho\, dv$,

$$Q = \int_0^1 \int_0^1 \int_0^1 30x^2y\, dx\, dy\, dz = 5\ \mu\text{C}$$

For the change in limits on y,

$$Q = \int_0^1 \int_{-1}^0 \int_0^1 30x^2y\, dx\, dy\, dz = -5\ \mu\text{C}$$

Problem-2

Three point charges, $Q_1 = 30$ nC, $Q_2 = 150$ nC, and $Q_3 = -70$ nC, are enclosed by surface S. What net flux crosses S?

Since electric flux was defined as originating on positive charge and terminating on negative charge, part of the flux from the positive charges terminates on the negative charge.

$$\Psi_{\text{net}} = Q_{\text{net}} = 30 + 150 - 70 = 110\ \text{nC}$$

Problem-3

A point charge, $Q = 30$ nC, is located at the origin in cartesian coordinates. Find the electric flux density $\mathbf{D}$ at $(1, 3, -4)$ m.

Referring to Fig. 3.12,

$$\mathbf{D} = \frac{Q}{4\pi R^2}\mathbf{a}_R$$

$$= \frac{30 \times 10^{-9}}{4\pi(26)}\left(\frac{\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{26}}\right)$$

$$= (9.18 \times 10^{-11})\left(\frac{\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{26}}\right)\ \text{C/m}^2$$

or, more conveniently, $D = 91.8$ pC/m^2.

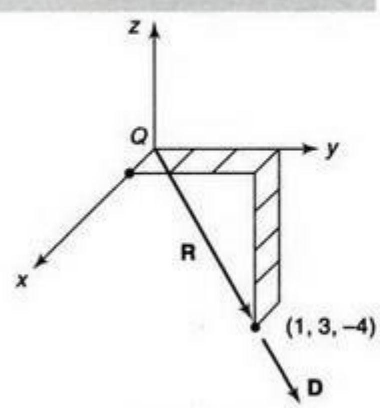


Fig. 3.12

Problem-4

Given that $\mathbf{D} = 10x\mathbf{a}_x$ (C/m^2), determine the flux crossing a 1-m^2 area that is normal to the x axis at $x = 3$ m.

Since $\mathbf{D}$ is constant over the area and perpendicular to it,

$$\Psi = DA = (30\ \text{C/m}^2)(1\ \text{m}^2) = 30\ \text{C}$$

Problem-5

Given the vector field $\mathbf{A} = 5x^2\left(\sin\frac{\pi x}{2}\right)\mathbf{a}_x$, find div $\mathbf{A}$ at $x = 1$.

$$\text{div } \mathbf{A} = \frac{\partial}{\partial x}\left(5x^2 \sin\frac{\pi x}{2}\right) = 5x^2\left(\cos\frac{\pi x}{2}\right)\frac{\pi}{2} + 10x \sin\frac{\pi x}{2} = \frac{5}{2}\pi x^2 \cos\frac{\pi x}{2} + 10x \sin\frac{\pi x}{2}$$

and $\text{div } \mathbf{A}\big|_{x=1} = 10.$

Problem-6

Given that $\mathbf{D} = (10r^3/4)\mathbf{a}_r$ (C/m^2) in the region $0 < r \le 3$ m in cylindrical coordinates and $\mathbf{D} = (810/4r)\mathbf{a}_r$ (C/m^2) elsewhere, find the charge density.

For $0 < r \le 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{10r^4}{4}\right) = 10r^2 \text{ (C/m}^3\text{)}$$

and for $r > 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}(810/4) = 0$$

Problem-7

In order to illustrate the application of (13) or (14), let us find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Fig. 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Since $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\,\text{V} \cdot \text{m}$, we may now use (13) or (14) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

Problem-8

Ex. *A charge $Q_1 = -20\mu C$ is located at P (– 6, 4, 6) and a charge $Q_2 = 50\mu C$ is located at R (5, 8, – 2) in a free space. Find the force exerted on Q_2 by Q_1 in vector form. The distances given are in metres.*

Sol. : From the co-ordinates of P and R , the respective position vectors are –

$$\bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$$

and

$$\bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$$

The force on Q_2 is given by,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P} = [5-(-6)]\,\bar{a}_x + (8-4)\,\bar{a}_y + [-2-(6)\bar{a}_z]$$

$$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$$

$$\therefore \quad |R_{12}| = \sqrt{(11)^2 + (4)^2 + (-8)^2} = 14.1774$$

UNIT - II:
Electrostatics-II

- Gauss Law and Applications
- Electric Potential
- Relations between E and V
- Maxwell's Equations for Electrostatic Fields
- Dielectric Constant
- Isotropic and Homogeneous Dielectrics
- Continuity Equation
- Relaxation Time
- Poisson's and Laplace's Equations
- Boundary conditions-conductor-Dielectric and Dielectric-Dielectric
- Problems.

Gauss's Law:

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

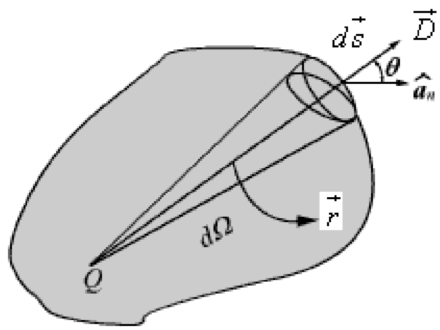


Fig 3: Gauss's Law

Let us consider a point charge *Q* located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance *r* on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2}ds\cos\theta$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of *Q*.

Therefore we can write $d\psi = \frac{Q}{4\pi}d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_s d\psi = \frac{Q}{4\pi}\oint_s d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law.

Hence we have,

$$Q_{enc} = \oint_S D \cdot ds = \int_V \rho_V \, dv$$

Applying Divergence theorem we have,

$$\oint_S D \cdot ds = \int_V \nabla \cdot D \, dv$$

Comparing the above two equations, we have

$$\int_V \nabla \cdot D \, dv = \int_V \rho_V \, dv$$

This equation is called the 1st Maxwell's equation of electrostatics.

Application of Gauss's Law:

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. $\vec{E}$ **due to an infinite line charge**

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density $_L$C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write, $\rho_l l = \varepsilon_0 E . 2\pi \rho l$

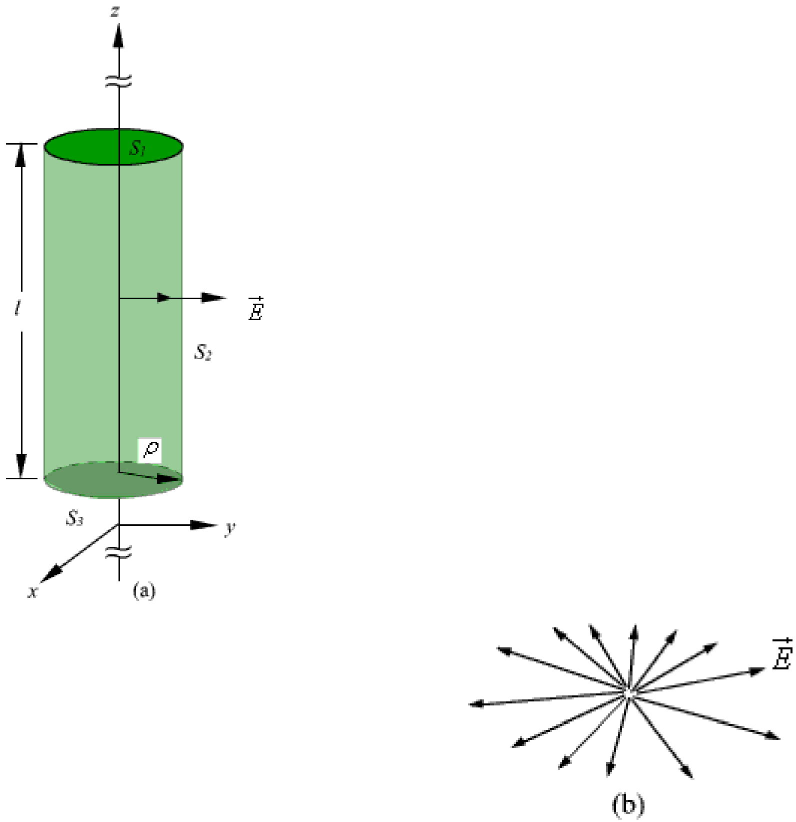


Fig 4: Infinite Line Charge

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 \rho}\hat{a}_\rho$$

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x-z* plane as shown in figure 5. Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D} \cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \qquad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

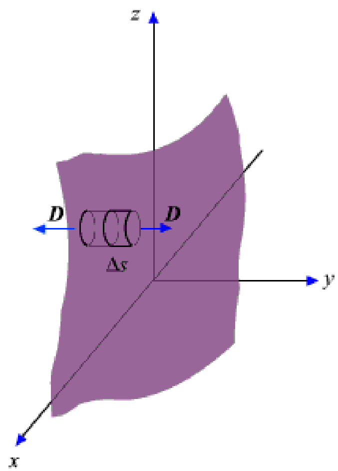


Fig 5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r0 having a uniform volume charge density of rv C/m3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius r < r0 and r > r0 as shown in Fig. 6 (a) and Fig. 6(b).

For the region $r \leq r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3$$

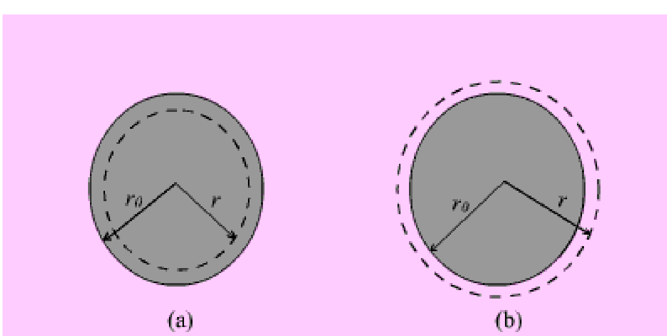


Fig 6: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$

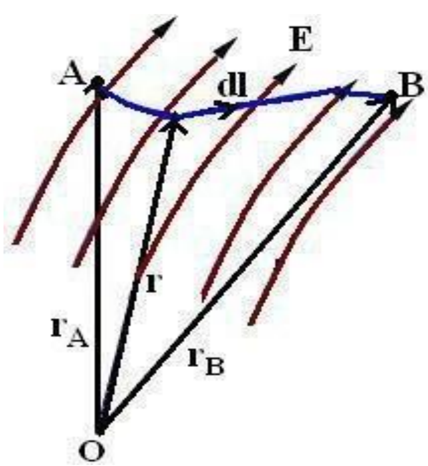
Electric Potential / Electrostatic Potential (V):

If a charge is placed in the vicinity of another charge (or in the field of another charge), it experiences a force. If a field being acted on by a force is moved from one point to another, then work is either said to be done on the system or by the system.

Say a point charge Q is moved from point A to point B in an electric field E, then the work done in moving the point charge is given as:

WA→B = - ∫AB (F . dl) = - Q ∫AB(E . dl)

where the – ve sign indicates that the work is done on the system by an external agent.



The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(VAB).

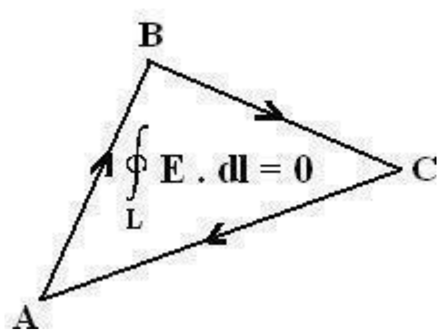
VAB = WA→B / Q

- ∫AB(E . dl)

- ∫InitialFinal (E . dl)

If the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

The electrostatic field is conservative i.e. the value of the line integral depends only on end points and is independent of the path taken.



- Since the electrostatic field is conservative, the electric potential can also be written as:

$$V_{AB} = -\int_A^B \bar{E} \cdot \bar{dl}$$

$$V_{AB} = -\int_A^{p_0} \bar{E} \cdot \bar{dl} - \int_{p_0}^B \bar{E} \cdot \bar{dl}$$

$$V_{AB} = -\int_{p_0}^B \bar{E} \cdot \bar{dl} + \int_{p_0}^A \bar{E} \cdot \bar{dl}$$

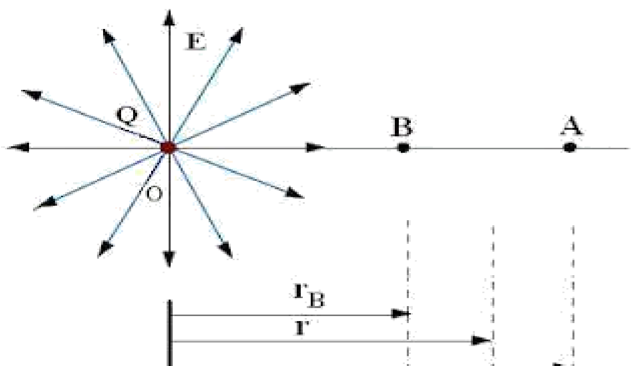
$$V_{AB} = V_B - V_A$$

Thus the potential difference between two points in an electrostatic field is a scalar field that is defined at every point in space and is independent of the path taken.

- The work done in moving a point charge from point A to point B can be written as:

$W_{A\rightarrow B} = - Q\,[V_B - V_A] = -Q\int_A^B \bar{E}\,\bar{dl}$

- Consider a point charge Q at origin O.



Now if a unit test charge is moved from point A to Point B, then the potential difference between them is given as:

$$V_{AB} = -\int_A^B \mathbf{E}\cdot d\mathbf{l} = -\int_{r_A}^{r_B} \mathbf{E}\cdot d\mathbf{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r\cdot dr\,\mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = V_B - V_A$$

- Electrostatic potential or Scalar Electric potential (V) at any point P is given by:

$$V = -\int_{P_0}^{P} \bar{E}\,\bar{dl}$$

The reference point Po is where the potential is zero (analogues to ground in a circuit). The reference is often taken to be at infinity so that the potential of a point in space is defined as

$$V = -\int_{\infty}^{P} \bar{E}\,d\bar{l}$$

Basically potential is considered to be zero at infinity. Thus potential at any point (rB = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge frominfinity to that point (i.e. rA → ∞)

Electric potential (V) at point r due to a point charge Q located at a point with position vector r1 is given as:

$$V = \frac{Q}{4\pi\varepsilon|r - r_1|}$$

Similarly for N point charges Q1, Q2 ….Qn located at points with position vectors r1, r2, r3…..rn, theelectric potential (V) at point r is given as:

$$V = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{N}\frac{Q_k}{|r - r_k|} \qquad V = \frac{Q}{4\pi\varepsilon r}$$

The charge element dQ and the total charge due to different charge distribution is given as:

dQ = ρldl → Q = ∫L (ρldl) → (Line Charge)

dQ = ρsds → Q = ∫S (ρsds) → (Surface Charge)

dQ = ρvdv → Q = ∫V (ρvdv) → (Volume Charge)

$$V = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon|r - r_1|} \qquad \textbf{(Line Charge)}$$

$$V = \int_S \frac{\rho_S\, ds}{4\pi\varepsilon|r - r_1|} \qquad \textbf{(Surface Charge)}$$

$$V = \int_V \frac{\rho_V\, dv}{4\pi\varepsilon|r - r_1|} \qquad \textbf{(Volume Charge)}$$

Second Maxwell's Equation of Electrostatics:

The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(V_{AB}).

$V_{AB} = V_B – V_A$

Similarly,

$V_{BA} = V_A – V_B$

Hence it‘s clear that potential difference is independent of the path taken. Therefore

$V_{AB} = - V_{BA}$

$V_{AB} + V_{BA} = 0$

$\int_{AB} (E . dl) + [- \int_{BA} (E . dl)] = 0$

$$\oint_L \mathbf{E} \cdot \mathbf{dl} = 0$$

The above equation is called the second Maxwell‘s Equation of Electrostatics in integral form.. The above equation shows that the line integral of Electric field intensity (E) along a closed path is equal to zero.
In simple words—No work is done in moving a charge along a closed path in an electrostatic field.

Applying Stokes‘ Theorem to the above Equation, we have:

$$\oint_L \mathbf{E} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{ds} = 0$$

$$\text{--->} \ \nabla \times \mathbf{E} = 0$$

If the Curl of any vector field is equal to zero, then such a vector field is called an Irrotational or Conservative Field. Hence an electrostatic field is also called a conservative field.
The above equation is called the second Maxwell‘s Equation of Electrostatics in differential form.

Relationship Between Electric Field Intensity (E) and Electric Potential (V):

Since Electric potential is a scalar quantity, hence dV (as a function of x, y and z variables) can be written as:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$\left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right)\cdot\left(dx\,a_x + dy\,a_y + dz\,a_z\right) = -E\,.\,dl$$

$$\nabla V\,.\,dl = -E\,.\,dl \quad \text{--->} \quad \boxed{E = -\nabla V}$$

Hence the Electric field intensity (E) is the negative gradient of Electric potential (V).
The negative sign shows that E is directed from higher to lower values of V i.e. E is opposite to the direction in which V increases.

Work Done To Assemble Charges:

In case, if we wish to assemble a number of charges in an empty system, work is required to do so. Also electrostatic energy is said to be stored in such a collection.

Let us build up a system in which we position three point charges Q1, Q2 and Q3 at position r1, r2 and r3 respectively in an initially empty system.

Consider a point charge Q1 transferred from infinity to position r1 in the system. It takes no work to bring the first charge from infinity since there is no electric field to fight against (as the system is empty i.e. charge free).

Hence, W1 = 0 J

Now bring in another point charge Q2 from infinity to position r2 in the system. In this case we have to do work against the electric field generated by the first charge Q1.

Hence, W2 = Q2 V21

where V21 is the electrostatic potential at point r2 due to Q1.

- Work done W2 is also given as:

$$W_2 = \frac{Q_2 Q_1}{4\pi\varepsilon\ |r_2 - r_1|}$$

Now bring in another point charge Q3 from infinity to position r3 in the system. In this case we have to do work against the electric field generated by Q1 and Q2.

Hence, W3 = Q3 V31 + Q3 V32 = Q3 (V31 + V32)

where V31 and V32 are electrostatic potential at point r3 due to Q1 and Q2 respectively.

The work done is simply the sum of the work done against the electric field generated by point charge Q1 and Q2 taken in isolation:

$$W_3 = \frac{Q_3 Q_1}{4\pi\varepsilon\,|r_3 - r_1|} + \frac{Q_3 Q_2}{4\pi\varepsilon\,|r_3 - r_2|}$$

- Thus the total work done in assembling the three charges is given as:

WE = W1 + W2 + W3

0 + Q2 V21 + Q3 (V31 + V32)

Also total work done (WE) is given as:

$$W_E = \frac{1}{4\pi\varepsilon}\left(\frac{Q_2 Q_1}{|r_2 - r_1|} + \frac{Q_3 Q_1}{|r_3 - r_1|} + \frac{Q_3 Q_2}{|r_3 - r_2|}\right)$$

If the charges were positioned in reverse order, then the total work done in assembling them is given as:

WE = W3 + W2+ W1
= 0 + Q2V23 + Q3(V12+ V13)

Where V23 is the electrostatic potential at point r2 due to Q3 and V12 and V13 are electrostatic potential at point r1 due to Q2 and Q3 respectively.

- Adding the above two equations we have,

2WE = Q1 (V12 + V13) + Q2 (V21 + V23) + Q3 (V31 + V32)

= Q1 V1 + Q2 V2 + Q3 V3

Hence

WE =1 / 2 [Q1V1 + Q2V2 + Q3V3]

where V1, V2 and V3 are total potentials at position r1, r2 and r3 respectively.

- The result can be generalized for N point charges as:

$$W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

The above equation has three interpretation: This equation represents the potential energy of the system.This is the work done in bringing the static charges from infinity and assembling them in the required system. This is the kinetic energy which would be released if the system gets dissolved i.e. the charges returns back to infinity.

In place of point charge, if the system has continuous charge distribution (line, surface or volume charge), then the total work done in assembling them is given as:

$$W_E = \frac{1}{2}\int_L \rho_L V\,dl \quad \text{(Line Charge)}$$

$$W_E = \frac{1}{2}\int_S \rho_S V\,ds \quad \text{(Surface Charge)}$$

$$W_E = \frac{1}{2}\int_V \rho_V V\,dv \quad \text{(Volume Charge)}$$

Since $\rho v = \nabla . D$ and $E = - \nabla V$,

Substituting the values in the above equation, work done in assembling a volume charge distribution in terms of electric field and flux density is given as:

$$W_E = \frac{1}{2}\int_V D \,.\, E \; dv \;=\; \frac{1}{2}\int_V \varepsilon E^2 \; dv$$

The above equation tells us that the potential energy of a continuous charge distribution is stored in an electric field.

The electrostatic energy density wE is defined as:

$$w_E = \frac{1}{2}\varepsilon E^2 \quad ; \qquad W_E = \int_V w_E \, dv$$

Properties of Materials and Steady Electric Current:

Electric field can not only exist in free space and vacuum but also in any material medium. When an electric field is applied to the material, the material will modify the electric field either by strengthening it or weakening it, depending on what kind of material it is.

Materials are classified into 3 groups based on conductivity / electrical property:

- Conductors (Metals like Copper, Aluminum, etc.) have high conductivity ($\sigma >> 1$).
- Insulators / Dielectric (Vacuum, Glass, Rubber, etc.) have low conductivity ($\sigma << 1$).
- Semiconductors (Silicon, Germanium, etc.) have intermediate conductivity.

Conductivity (σ) is a measure of the ability of the material to conduct electricity. It is the reciprocal of resistivity (ρ). Units of conductivity are Siemens/meter and mho.

The basic difference between a conductor and an insulator lies in the amount of free electrons available for conduction of current. Conductors have a large amount of free electrons where as insulators have only a few number ofelectrons for conduction of current. Most of the conductors obey ohm's law. Such conductors are also called ohmic conductors.

Due to the movement of free charges, several types of electric current can be caused. The different types of electric current are:

- Conduction Current.
- Convection Current.
- Displacement Current.

Electric current:

Electric current (I) defines the rate at which the net charge passes through a wire of cross sectional surface area S.

Mathematically,

If a net charge ΔQ moves across surface S in some small amount of time Δt, electric current(I) is defined as:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

How fast or how speed the charges will move depends on the nature of the material medium.

Current density:

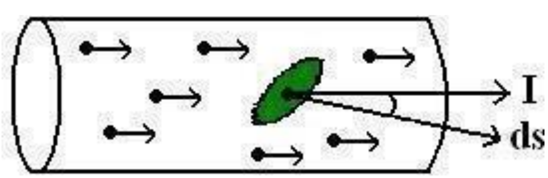
Current density (J) is defined as current ΔI flowing through surface ΔS.

Imagine surface area ΔS inside a conductor at right angles to the flow of current. As the area approaches zero, the current density at a point is defined as:

$$J = \lim_{\Delta s \to 0} \frac{\Delta I}{\Delta S}$$

The above equation is applicable only when current density (J) is normal to the surface.

In case if current density(J) is not perpendicular to the surface, consider a small area ds of the conductor at an angle θ to the flow of current as shown:



In this case current flowing through the area is given as:

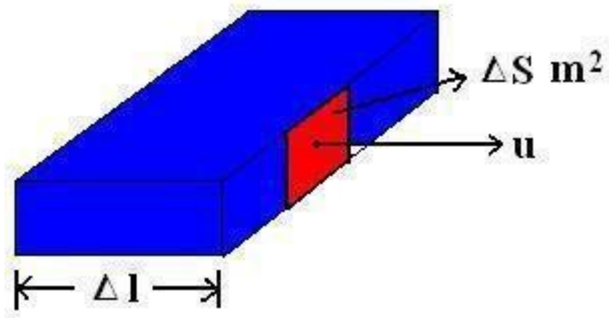
$dI = J\ dS \cos\theta = J \cdot dS$ and $I = \int_S \bar{J} \cdot d\bar{s}$

Where angle θ is the angle between the normal to the area and direction of the current.

From the above equation it‘s clear that electric current is a scalar quantity.

CONVECTION CURRENT DENSITY:

Convection current occurs in insulators or dielectrics such as liquid, vacuum and rarified gas. Convection current results from motion of electrons or ions in an insulating medium. Since convection current doesn‘t involve conductors, hence it does not satisfy ohm‘s law. Consider a filament where there is a flow of charge ρv at a velocity u = uy ay.



- Hence the current is given as:

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

But we know $\boxed{\Delta Q = \rho_V \Delta V}$

Hence

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_V \Delta V}{\Delta t} = \rho_V \Delta S \frac{\Delta l}{\Delta t}$$

$$= \rho_V \Delta S\, u_y$$

Again, we also know that $\boxed{J_y = \frac{\Delta I}{\Delta S}}$

Hence $$J_y = \frac{\Delta I}{\Delta S} = \rho_V u_y$$

Where uy is the velocity of the moving electron or ion and ρ_v is the free volume charge density.

- Hence the convection current density in general is given as:

$$J = \rho_v u$$

Conduction Current Density:

Conduction current occurs in conductors where there are a large number of free electrons. Conduction current occurs due to the drift motion of electrons (charge carriers). Conduction current obeys ohm‘s law.
When an external electric field is applied to a metallic conductor, conduction current occurs due to the drift of electrons.

The charge inside the conductor experiences a force due to the electric field and hence should accelerate but due to continuous collision with atomic lattice, their velocity is reduced. The net effect is that the electrons moves or drifts with an average velocity called the drift velocity (υd) which is proportional to the applied electric field (E).

Hence according to Newton‘s law, if an electron with a mass m is moving in an electric field E with anaverage drift velocity υd, the the average change in momentum of the free electron must be equal to the applied force (F = - e E).

$$\frac{m\upsilon_d}{\tau} = -eE$$

where τ is the average time interval between collision.

$$\upsilon_d = \left(-\frac{e\tau}{m}\right)E$$

The drift velocity per unit applied electric field is called the mobility of electrons (μe).

$\upsilon d = - \mu e\, E$

where μe is defined as:

$$\mu_e = \left(-\frac{e\tau}{m}\right)$$

Consider a conducting wire in which charges subjected to an electric field are moving with drift velocity υ_d.
Say there are Ne free electrons per cubic meter of conductor, then the free volume charge density(ρ_v)within the wire is

$\rho_v = -e N_e$

The charge ΔQ is given as:

$\Delta Q = \rho_v \Delta V = -e N_e \Delta S \Delta l = -e N_e \Delta S \upsilon_d \Delta t$

- The incremental current is thus given as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = -N_e e \Delta S \upsilon_d$$

Now since $\boxed{\upsilon_d = -\mu_e E}$

Therefore

$$\Delta I = N_e e \Delta S \mu_e E$$

The conduction current density is thus defined as:

$$J_c = \frac{\Delta I}{\Delta S} = N_e e \mu_e E = \sigma E$$

where σ is the conductivity of the material.

The above equation is known as the Ohm‘s law in point form and is valid at every point in space.

In a semiconductor, current flow is due to the movement of both electrons and holes, hence conductivity is given as:

$\sigma = (N_e \mu_e + N_h \mu_h) e$

DIELECTRC CONSTANT:

It is also known as Relative permittivity.

If two charges q 1 and q 2 are separated from each other by a small distance r. Then by using the coulombs law of forces the equation formed will be

$$\mathbf{F}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{\mathbf{r}^2}$$

In the above equation ε_0 is the electrical permittivity or you can say it, Dielectric constant.

If we repeat the above case with only one change i.e. only change in the separation medium between the charges. Here some material medium must be used. Then the equation formed will be.

$$\mathbf{F}_m = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{\mathbf{r}^2}$$

Now after division of above two equations

$$\frac{\mathbf{F}_0}{\mathbf{F}_m} = \frac{\varepsilon}{\varepsilon_0} = \varepsilon_r \text{ Or } k$$

In the above figure

$\varepsilon_\mathbf{r}$ is the Relative Permittivity. Again one thing to notice is that the dielectric constant is represented by the symbol (K) but permittivity by the symbol $\varepsilon_\mathbf{r}$

CONTINUITY EQUATION:

The continuity equation is derived from two of Maxwell's equations. It states that the divergence of the current density is equal to the negative rate of change of the charge density,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

Derivation

One of Maxwell's equations, Ampère's law, states that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking the divergence of both sides results in

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t},$$

but the divergence of a curl is zero, so that

$$\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = 0. \qquad (1)$$

Another one of Maxwell's equations, Gauss's law, states that

$$\nabla \cdot \mathbf{D} = \rho.$$

Substitute this into equation (1) to obtain

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

which is the continuity equation.

1.13 RELAXATION TIME:

- Let us consider that a charge is introduced at some interior point of a given material (conductor or dielectric).
- From, continuity of current equation, we have

$$\overline{J} = \frac{-rf_v}{rt} ----(1)$$

- We have, the point form of Ohm's law as,

$$\overline{J} = 6\overline{E} ---(2)$$

- From Gauss's law, we have,

$$\nabla\overline{D} = f_v \Rightarrow \in \nabla.\overline{E} = f_v \left[\because \overline{D} = \in.\overline{E}\right]$$

$$\therefore \nabla.\overline{E} = \frac{f_v}{\in} ------(1)$$

- Substitute equations (2) and (3) in equation (1), we get

$$\nabla.6\overline{E}f = 6.\nabla.\overline{E} = 6.\frac{f_v}{\in} = \frac{-\partial f_v}{\partial t}$$

$$\Rightarrow \frac{\partial f_v}{\partial t} + \frac{6}{\in}.f_v = 0 ------(4)$$

- The above equation is a homogeneous linear ordinary differential equation. By separating variable in eq (4), we get,

$$\frac{\partial f_v}{\partial t} = \frac{-6}{\in}.f_v$$

$$\Rightarrow \frac{\partial f_v}{\partial t} = \frac{-6}{\in}.\partial t$$

- Now integrate on both sides of above equation

$$\int \frac{\partial f_v}{\partial t} = -\frac{6}{\in}.\int \partial t$$

$$\Rightarrow \ln f_v = -\frac{6}{\in}t + \ln f_{v0}$$

Where $\ln p_{v0}$ is a constant of integration.

Thus,

$$\boxed{f_v = f_{v0}e^{-t/Tr}} \quad -----(5)$$

$$\boxed{T_r = \frac{\in}{6}}$$

- In eq (5), f_{v0} is the initial charge density (i.e fv at t=0).
- We can see from the equation that as a result of introducing charge at some interior point of the material there is a decay of volume charge density f_v.
- The time constant "T_r" is known as the relaxation time or rearrangement time.
- Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} = 36.8 percent f its initial value.
- The relation time is short for good conductors and long for good dielectrics.

LAPLACE'S AND POISSON'S EQUATIONS:

A useful approach to the calculation of electric potentials is to relate that potential to the charge density which gives rise to it. The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

E = electric field
ρ = charge density
ε_0 = permittivity

and the electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V$$

Therefore the potential is related to the charge density by Poisson's equation

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

In a charge-free region of space, this becomes LaPlace's equation

$$\nabla^2 V = 0$$

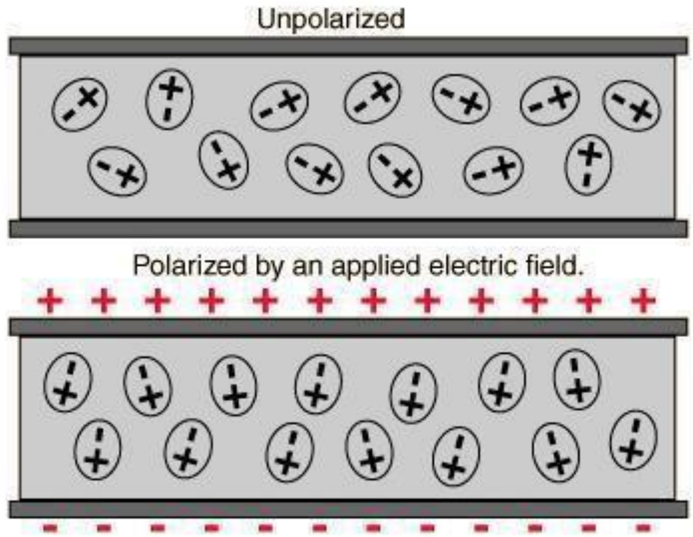
This mathematical operation, the divergence of the gradient of a function, is called the LaPlacian. Expressing the LaPlacian in different coordinate systems to take advantage of the symmetry of a charge distribution helps in the solution for the electric potential V. For example, if the charge distribution has spherical symmetry, you use the LaPlacian in spherical polar coordinates.

Since the potential is a scalar function, this approach has advantages over trying to calculate the electric field directly. Once the potential has been calculated, the electric field can be computed by taking the gradient of the potential.

Polarization of Dielectric:

If a material contains polar molecules, they will generally be in random orientations when no electric field is applied. An applied electric field will polarize the material by orienting the dipole moments of polar molecules.

This decreases the effective electric field between the plates and will increase the capacitance of the parallel plate structure. The dielectric must be a good electric insulator so as to minimize any DC leakage current through a capacitor.



The presence of the dielectric decreases the electric field produced by a given charge density.

$$E_{\text{effective}} = E - E_{\text{polarization}} = \frac{\sigma}{k\varepsilon_0}$$

The factor k by which the effective field is decreased by the polarization of the dielectric is called the dielectric constant of the material.

Solved problems:

Problem1:

Three point charges, $Q_1 = 30$ nC, $Q_2 = 150$ nC, and $Q_3 = -70$ nC, are enclosed by surface S. What net flux crosses S?

Since electric flux was defined as originating on positive charge and terminating on negative charge, part of the flux from the positive charges terminates on the negative charge.

$$\Psi_{net} = Q_{net} = 30 + 150 - 70 = 110 \text{ nC}$$

Problem-2

An electrostatic field is given by $\mathbf{E} = (x/2 + 2y)\mathbf{a}_x + 2x\mathbf{a}_y$ (V/m). Find the work done in moving a point charge $Q = -20\ \mu$C (a) from the origin to (4, 0, 0) m, and (b) from (4, 0, 0) m to (4, 2, 0) m.

(a) The first path is along the x axis, so that $d\mathbf{l} = dx\ \mathbf{a}_x$.

$$dW = -Q\mathbf{E}\cdot d\mathbf{l} = (20 \times 10^{-6})\left(\frac{x}{2} + 2y\right)dx$$

$$W = (20 \times 10^{-6})\int_0^4\left(\frac{x}{2} + 2y\right)dx = 80\ \mu\text{J}$$

(b) The second path is in the $\mathbf{a}_y$ direction, so that $d\mathbf{l} = dy\mathbf{a}_y$.

$$W = (20 \times 10^{-6})\int_0^2 2x\, dy = 320\ \mu\text{J}$$

Problem-3

What electric field intensity and current density correspond to a drift velocity of 6.0×10^{-4} m/s in a silver conductor?

For silver $\sigma = 61.7$ MS/m and $\mu = 5.6 \times 10^{-3}$ m²/V · s.

$$E = \frac{U}{\mu} = \frac{6.0 \times 10^{-4}}{5.6 \times 10^{-3}} = 1.07 \times 10^{-1} \text{ V/m}$$

$$J = \sigma E = 6.61 \times 10^6 \text{ A/m}^2$$

Problem-4

Find the current in the circular wire shown in Fig. 6.6 if the current density is $\mathbf{J} = 15(1 - e^{-1000r})\mathbf{a}_z$ (A/m²). The radius of the wire is 2 mm.

A cross section of the wire is chosen for S. Then

$$dI = \mathbf{J}\cdot d\mathbf{S}$$
$$= 15(1 - e^{-1000r})\mathbf{a}_z \cdot r\, dr\, d\phi\mathbf{a}_z$$

and

$$I = \int_0^{2\pi}\int_0^{0.002} 15(1 - e^{-1000r})r\, dr\, d\phi$$
$$= 1.33 \times 10^{-4} \text{ A} = 0.133 \text{ mA}$$

Any surface S which has a perimeter that meets the outer surface of the conductor all the way around will have the same total current, $I = 0.133$ mA, crossing it.

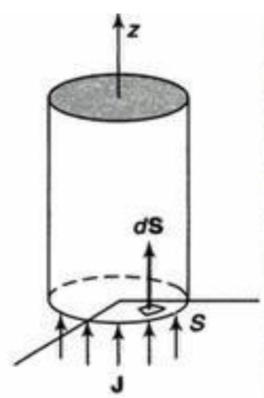


Fig. 6.6

Problem-5

Determine the relaxation time for silver, given that $\sigma = 6.17 \times 10^7$ S/m. If charge of density ρ_0 is placed within a silver block, find ρ after one, and also after five, time constants.

Since $\varepsilon \approx \varepsilon_0$,

$$\tau = \frac{\varepsilon}{\sigma} = \frac{10^{-9} 36\pi}{6.17 \times 10^7} = 1.43 \times 10^{-19} \text{ s}$$

Therefore

$$\text{at} \quad t = \tau: \qquad \rho = \rho_0 e^{-1} = 0.368\rho_0$$

$$\text{at} \quad t = 5\tau: \qquad \rho = \rho_0 e^{-5} = 6.74 \times 10^{-3}\rho_0$$

Problem-6

Find the magnitudes of **D** and **P** for a dielectric material in which $E = 0.15$ MV/m and $\chi_e = 4.25$.

Since $\varepsilon_r = \chi_e + 1 = 5.25$,

$$D = \varepsilon_0 \varepsilon_r E = \frac{10^{-9}}{36\pi}(5.25)(0.15 \times 10^6) = 6.96 \ \mu\text{C/m}^2$$

$$P = \chi_e \varepsilon_0 E = \frac{10^{-9}}{36\pi}(4.25)(0.15 \times 10^6) = 5.64 \ \mu\text{C/m}^2$$

UNIT-III

MAGNETOSTATICS

Contents:

- Biot - Savart's Law
- Ampere's Circuital Law and Applications
- Magnetic Flux Density
- Maxwell's Equations for Magnetostatic Fields
- Magnetic Scalar and Vector Potentials
- Ampere's Force Law
- Faraday's Law
- Transformer EMF
- Displacement Current Density
- Maxwell's Equations for time varying fields
- Illustrative Problems.

Introduction:

In previous chapters we have seen that an electrostatic field is produced by static or stationary charges. The relationship of the steady magnetic field to its sources is much more complicated.

The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later.

There are two major laws governing the magneto static fields are:

- Biot-Savart Law
- Ampere's Law

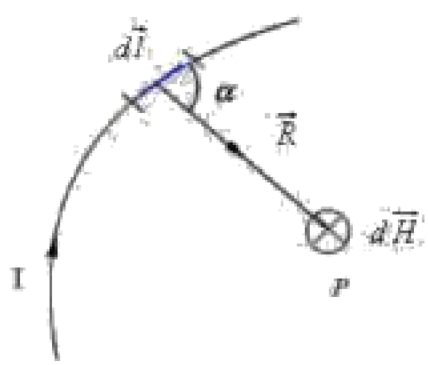
Usually, the magnetic field intensity is represented by the vector $\vec{H}$. It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 2.1.

$\vec{H}$ (or l) out of the page $\vec{H}$ (or l) into the page

Fig. Representation of magnetic field (or current)

Biot- Savart's Law:

This law relates the magnetic field intensity dH produced at a point due to a differential current element $Id\vec{l}$ as shown in Fig.



The magnetic field intensity $d\vec{H}$ at P can be written as,

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$

$$dH = \frac{IdlSin\alpha}{4\pi R^2}$$

where $R = \left|\vec{R}\right|$ is the distance of the current element from the point P.
The value of the constant of proportionality 'K' depends upon a property called permeability of the medium around the conductor. Permeability is represented by symbol 'm' and the constant 'K' is expressed in terms of 'm' as

Thus

$$\mathbf{dB} = \frac{\mu}{4\pi} \frac{\mathbf{I\ dl\ \sin\theta}}{\mathbf{r^2}}$$

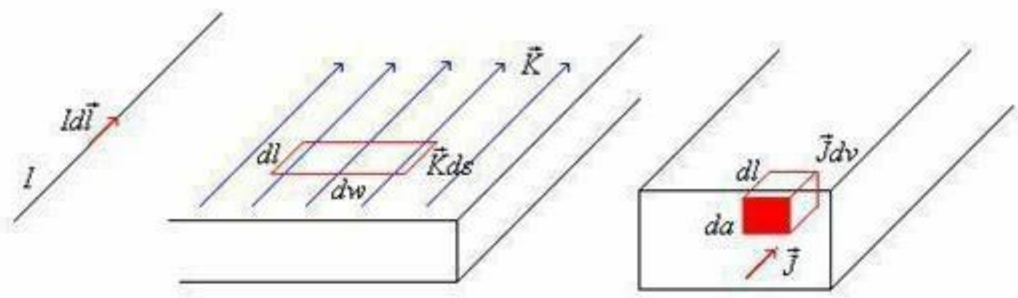
Magnetic field 'B' is a vector and unless we give the direction of 'dB', its description is not complete. Its direction is found to be perpendicular to the plane of 'r' and 'dl'.

If we assign the direction of the current 'I' to the length element 'dl', the vector product dl x r has magnitude r dl sinq and direction perpendicular to 'r' and 'dl'.

Hence, Biot–Savart law can be stated in vector form to give both the magnitude as well as direction of magnetic field due to a current element as

$$\overrightarrow{\mathbf{dB}} = \frac{\mu}{4\pi} \frac{\mathbf{I}\ (\overrightarrow{\mathbf{dl}} \times \overrightarrow{\mathbf{r}})}{\mathbf{r^3}}$$

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 2.3.



Line Current Surface Current Volume Current

Fig. 2.3: Different types of current distributions

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m2) we can write:

$$Id\vec{l} = \vec{K}ds = \vec{J}dv$$

(It may be noted that $I = Kdw = Jda$)

Employing Biot -Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions as

$$\vec{H} = \int_l \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$ .... for line current...........................

$$\vec{H} = \int_s \frac{Kd\vec{s} \times \vec{R}}{4\pi R^3}$$ .... for surface current ....

$$\vec{H} = \int_v \frac{\vec{J}dv \times \vec{R}}{4\pi R^3}$$ .... for volume current.....................

$\boldsymbol{H}$ Due to infinitely long straight conductor:

We consider a finite length of a conductor carrying a current $\vec{I}$ placed along z-axis as shown in the Fig 2.4. We determine the magnetic field at point P due to this current carrying conductor.

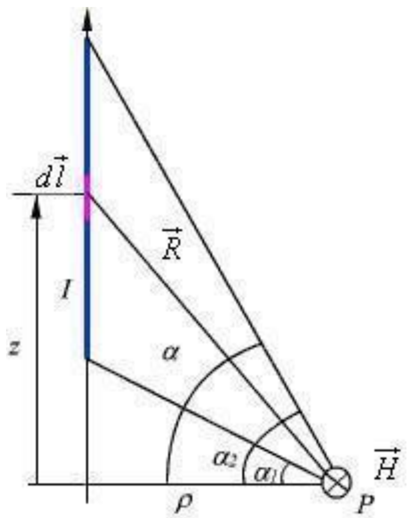


Fig. 2.4: Field at a point P due to a finite length current carrying conductor

With reference to Fig. 2.4, we find that

$$d\vec{l} = dz\,\hat{a}_z \quad and \quad \vec{R} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$

Applying Biot - Savart's law for the current element $\vec{I}\,d\vec{l}$ We can write,

$$\overrightarrow{dH} = \frac{Id\vec{l}\times\vec{R}}{4\pi R^3} = \frac{\rho dz\hat{a}_\phi}{4\pi[\rho^2+z^2]^{3/2}}$$

Substituting $\frac{z}{\rho} = \tan\alpha$ we can write,

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I}{4\pi}\frac{\rho^2\sec^2\alpha d\alpha}{\rho^3\sec^3\alpha}\hat{a}_\phi = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\phi$$

We find that, for an infinitely long conductor carrying a current I , $\alpha_2 = 90^0$ and $\alpha_1 = -90^0$ Therefore

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

Ampere's Circuital Law:

Ampere's circuital law states that the line integral of the magnetic field $\vec{H}$ (circulation of H) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H}.d\vec{l} = I_{enc}$$

The total current I enc can be written as,

$$I_{enc} = \int_S \vec{J}.d\vec{s}$$

By applying Stoke's theorem, we can write

$$\oint \vec{H}.d\vec{l} = \int_S \nabla \times \vec{H}.d\vec{s}$$

$$\therefore \int_S \nabla \times \vec{H}.d\vec{s} = \int_S \vec{J}.d\vec{s}$$

$$\therefore \nabla \times \vec{H} = \vec{J}$$

Which is the Ampere's circuital law in the point form and Maxwell's equation for magneto static fields.

Applications of Ampere's circuital law:

1 It is used to find H and B due to any type of current distribution.
2 If H or B is known then it is also used to find current enclosed by any closed path.

We illustrate the application of Ampere's Law with some examples.

H Due to infinitely long straight conductor :(using Ampere's circuital law)

We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 2.5. Using Ampere's Law, we consider the close path to be a circle of radius ρ as shown in the Fig. 4.5.

If we consider a small current element $Id\vec{l}(= Idz\hat{a}_z)$, $d\vec{H}$ is perpendicular to the plane containing both $d\vec{l}$ and $\vec{R}(= \rho\hat{a}_\rho)$. Therefore only component of $\vec{H}$ that will be present is H_ϕ,i.e., $\vec{H} = H_\phi\hat{a}_\phi$.

By applying Ampere's law we can write,

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho 2\pi = I$$

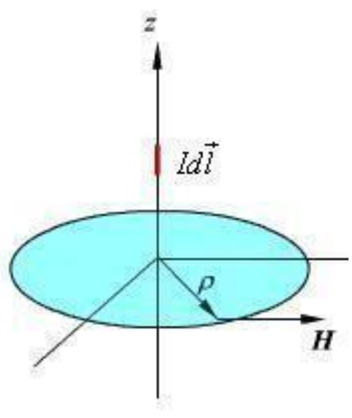


Fig. Magnetic field due to an infinite thin current carrying conductor

H Due to infinitely long coaxial conductor :(using Ampere's circuital law)

We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 2.6. We compute the magnetic field as a function of ρ as follows:

In the region $0 \leq \rho \leq R_1$

$$I_{enc} = I\frac{\rho^2}{R_1^2}$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho} = \frac{I\rho}{2\pi a^2}$$

In the region $R_1 \leq \rho \leq R_2$

$$I_{enc} = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

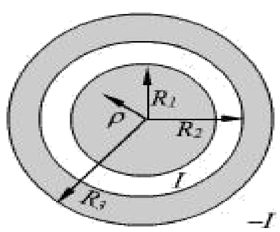


Fig. 2.6: Coaxial conductor carrying equal and opposite currents in the region

$R_2 \leq \rho \leq R_3$

$$H_\phi = \frac{I}{2\pi\rho}\frac{R_3^2 - \rho^2}{R_3^2 - R_2^2}$$

In the region $\rho > R_3$

$$I_{nec} = 0 \qquad H_\phi = 0$$

Magnetic Flux Density:

In simple matter, the magnetic flux density $\vec{B}$ related to the magnetic field intensity $\vec{H}$ as $\vec{B} = \mu\vec{H}$ where μ called the permeability. In particular when we consider the free space $\vec{B} = \mu_0\vec{H}$ where $\mu_0 = 4\pi\times10^{-7}$ H/m is the permeability of the free space. Magnetic flux density is measured in terms of Wb/m 2 .

The magnetic flux density through a surface is given by:

$$\psi = \int_S \vec{B}.d\vec{s} \quad \text{Wb}$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 6 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.

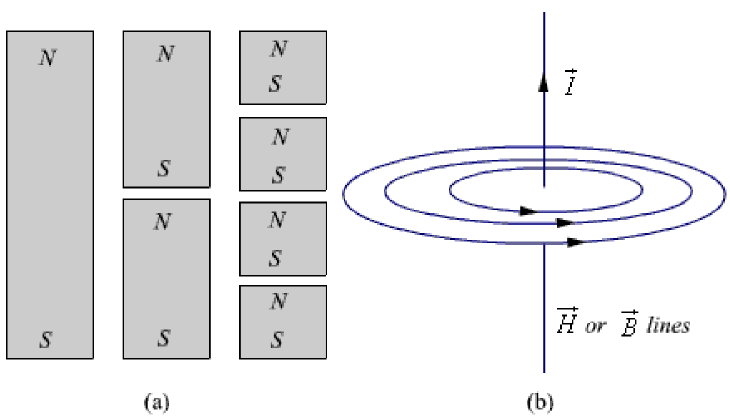


Fig. 6: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor

Maxwell's 2nd equation for static magnetic fields:

Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 6 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

$\oint_S \vec{B}.d\vec{s} = 0$ ....in integral form

which is the Gauss's law for the magnetic field.

By applying divergence theorem, we can write:

$$\oint_S \vec{B}.d\vec{s} = \int_V \nabla.\vec{B}dv = 0$$

Hence,........ $\nabla.\vec{B} = 0$ .... in point/differential form

which is the Gauss's law for the magnetic field in point form.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a scalar magnetic potential and write:

$\vec{H} = -\nabla V_m$

From Ampere's law , we know that

$\nabla \times \vec{H} = \vec{J}$

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$.

Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, Vm in general is not a single valued function of position. This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 7.

In the region $a < \rho < b$, $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

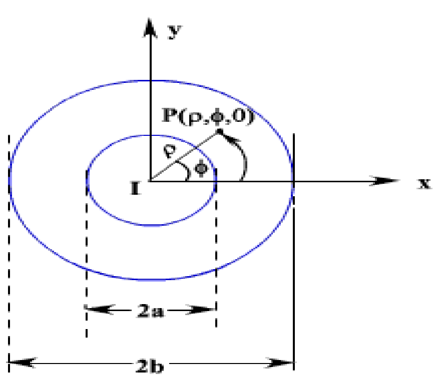


Fig. 7: Cross Section of a Coaxial Line

If Vm is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

If we set Vm = 0 at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but Vm this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of Vm keeps changing as we complete additional laps to pass through the same point. We introduced Vm analogous to electostatic potential V.

But for static electric fields,

$\nabla \times \vec{E} = 0$ and $\oint \vec{E} \cdot d\vec{l} = 0$

whereas for steady magnetic field $\nabla \times \vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H} \cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the vector magnetic potential which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla . \vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla . \left(\nabla \times \vec{A}\right) = 0$, we can write $\vec{B} = \nabla \times \vec{A}$.

Here, the vector field is $\vec{A}$ called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation. We have introduced the vector function and $\vec{B}$ related its $\vec{A}$ curl to . A vector $\vec{B}$ function is defined fully in terms of its curl as well as divergence. The choice of $\nabla . \vec{A}$ is made as follows.

$$\nabla \times \nabla \times \vec{A} = \mu \nabla \times \vec{H} = \mu \vec{J}$$

By using vector identity, $\nabla\times\nabla\times\vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$

$$\nabla(\nabla.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J}$$

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$.

Putting $\nabla.\vec{A} = 0$, we get $\nabla^2\vec{A} = -\mu\vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x = -\mu J_x$$.

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho}{R}dv', \qquad R = \left|\vec{r} - \vec{r}'\right|$$

$$\nabla.\vec{A} = \mu\varepsilon\frac{\partial V}{\partial t}$$

In case of time varying fields we shall see that , which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi}\int_{V'}\frac{J_x}{R}dv'$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}}{R}dv'$$

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$.

Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{L'}\frac{I}{R}d\vec{l}'$$

$$\vec{A} = \frac{\mu}{4\pi} \int_{s} \frac{\vec{K}}{R} ds'$$

.

The magnetic flux ψ through a given area S is given by

$$\psi = \int_{s} \vec{B}.d\vec{s}$$

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_{s} \nabla \times \vec{A}.d\vec{s} = \oint_{c} \vec{A}.d\vec{l}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Forces due to magnetic fields

There are three ways in which the force due to magnetic fields can be experienced. The force can be

(a) Force on a charged particle:

We have F_e=QE

This shows that if Q is positive, F_e and E are in same direction. It is found that the magnetic force F_m experienced by a charge Q moving with a velocity u in magnetic field B is

F_m=Qu x B

For a moving change Q in the presence of both electric and magnetic fields, the total force on the charge is given by

F = F_e+F_m

or

F=Q(E+u x B)

This is known as Lorentz force equation.

(b) Force on a current element:

To determine the force on a current element Idl of a current carrying conductor due to the magnetic field B, we take the equation

J=P_e u

We have Idl= $\frac{dQ}{dt.}.dl = dQ = \frac{dl}{dt} = dQ.u$

Hence

Idl= dQ.u

This shows that an elemental charge dQ moving with velocity u (thereby producing convection current element dQu) is equivalent to a conduction current element Idl. Thus the force on current element is give by

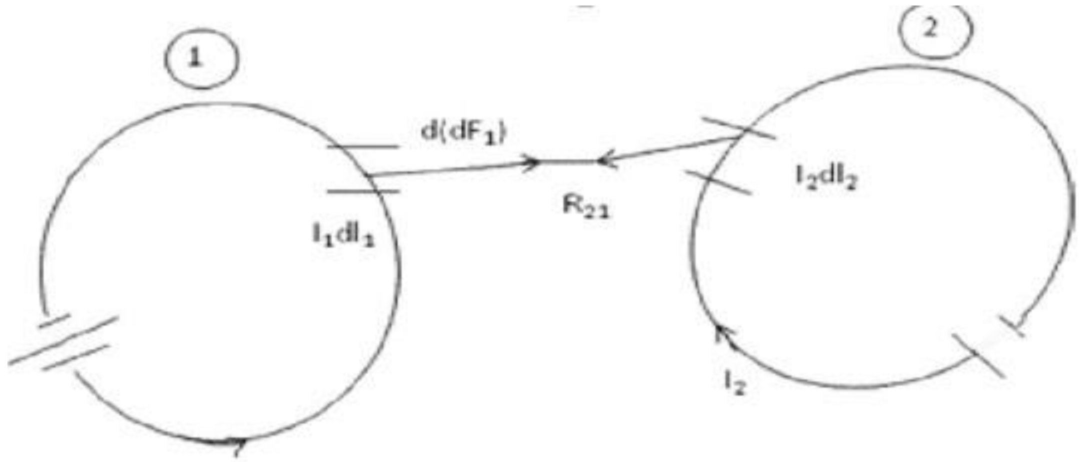
dF= Idl x B

If the current I is through a closed path L or circuit, the force on the circuit is given by

F= $\oint_L Idl \times B$

(c) Force between two current elements:

Consider the force between two elements I_1dl_1 and I_2dl_2. According to biotsavarts law, both current elements produce magnetic fields. Force $d(dF_1)$ on element I_1dl_1 due to field dB_2 produced by element I_2 dl_2 as shown in figure below:



$d(dF_1) = I_1 Dl_1 \times dB_2$

But from biot Savarts law

$$dB_2 = \frac{\mu_0 I_2 dl_2 \times a_{R21}}{4\pi R_{21}^2}$$

Hence

$$d(dF_1) = \frac{\mu_0 I_1 dl_1 \times (I_2 dl_2 \times a_{R21})}{4\pi R_{21}^2}$$

This equation is the law of force between two current elements.

We have $F1 = \frac{\mu_0 I_1 I_2 \times a_{R21}}{4\pi} \oint_{L_1}\oint_{L_2} \frac{dl_1 \times (dl_2 \times aR_{21})}{R_{21}^2}$

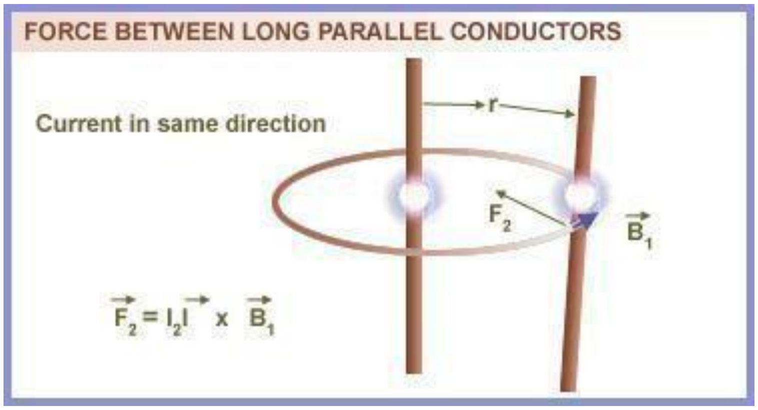
Faraday's Law:

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law.

Any change in the magnetic environment of a coil of wire will cause a voltage (emf) to be "induced" in the coil. No matter how the change is produced, the voltage will be generated. The change could be produced by changing the magnetic field strength, moving a magnet toward or away from the coil, moving the coil into or out of the magnetic field, rotating the coil relative to the magnet, etc.

Faraday's law is a fundamental relationship which comes from Maxwell's equations. It serves as a succinct summary of the ways a voltage (or emf) may be generated by a changing magnetic environment. The induced emf in a coil is equal to the negative of the rate of change of magnetic flux times the number of turns in the coil. It involves the interaction of charge with magnetic field.

When two current carrying conductors are placed next to each other, we notice that each induces a force on the other. Each conductor produces a magnetic field around itself (Biot– Savart law) and the second experiences a force that is given by the Lorentz force.



Mathematically, the induced emf can be written as

$$\text{Emf} = -\frac{d\phi}{dt} \quad \text{Volts}$$

where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (7) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$\text{Emf} = -N\frac{d\phi}{dt} \quad \text{Volts}$$

By defining the total flux linkage as

$$\lambda = N\phi$$

The emf can be written as

Emf = $-\dfrac{d\lambda}{dt}$

Continuing with equation (3), over a closed contour 'C' we can write

Emf = $\oint_C \vec{E}.d\vec{l}$

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_S \vec{B}.d\vec{s}$$

Where S is the surface for which 'C' is the contour.

From (11) and using (12) in (3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B}.d\vec{s}$$

By applying stokes theorem

$$\int_S \nabla\times\vec{E}.d\vec{s} = -\int_S \frac{\partial\vec{B}}{\partial t}.d\vec{s}$$

Therefore, we can write

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

which is the Faraday's law in the point form

We have said that non zero $\dfrac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf .

Displacement Current Density:

The equation

$\Delta \times H = J$ For static EM fields is modified to Modified to

$\Delta \times H = J + J_d$ ________________ (3.19)

To make the Ampere's law compatible for varying fields.

Now, applying divergence, we get

$$\Delta.(\Delta \times H) = 0 = \Delta.J + \Delta J_d$$
$$\Delta.J_d = -\Delta.J = \frac{de_v}{dt}$$

From Gauss Law, we have

$$e_v = \Delta.D$$

Therefore,

$$\Delta.J_d = \frac{d(\Delta.D)}{dt} = \Delta.\frac{dD}{dt}$$

$\Rightarrow J_d = \dfrac{dD}{dt}$ __________ (3.20)

MAXWELL’S EQUATIONS (Time varying Fields)

Introduction:

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla\times\vec{E}=0 \quad (1)$$

$$\nabla\cdot\vec{D}=\rho_v \quad (2)$$

For a linear and isotropic medium,

$$\vec{D}=\varepsilon\vec{E} \quad (3)$$

Similarly for the magnetostatic case

$$\nabla.\vec{B}=0 \quad (4)$$

$$\nabla\times\vec{H}=\vec{J} \quad (5)$$

$$\nabla\times\vec{H}=\vec{J} \quad (6)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field. Because of their concise statement, they embody a high level of mathematical sophistication and are therefore not generally introduced in an introductory treatment of the subject, except perhaps as summary relationships.

These basic equations of electricity and magnetism can be used as a starting point for advanced courses, but are usually first encountered as unifying equations after the study of electrical and magnetic phenomena.

Symbols Used		
E = Electric field	ρ = charge density	i = electric current
B = Magnetic field	ε0 = permittivity	J = current density
D = Electric displacement	μ0 = permeability	c = speed of light
H = Magnetic field strength	M = Magnetization	P = Polarization

Integral form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$

Gauss' law for magnetism $\oint \vec{B} \cdot d\vec{A} = 0$

III. Faraday's law of induction $\oint \vec{E} \cdot \vec{ds} = -\frac{d\Phi_B}{dt}$

IV. Ampere's law

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 i + \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E} \cdot d\vec{A}$$

Differential form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k\rho$

Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\ E = -\frac{\partial B}{\partial t}$

IV. Ampere's law

$$\nabla\ x\ B = \frac{4\pi k}{c^2} J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$= \frac{J}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$k = \frac{1}{4\pi\varepsilon_0} =$ *Coulomb's constant* $\quad c^2 = \frac{1}{\mu_0\varepsilon_0}$

Differential form with magnetic and/or polarizable media:

I. Gauss' law for electricity $\nabla \cdot D = \rho$

$$D = \varepsilon_0 E + P \quad \textit{General case}$$

$$D = \varepsilon_0 E \quad \textit{Free space}$$

$$D = \varepsilon E \quad \textit{Isotropic linear}$$

II. Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\dfrac{\partial B}{\partial t}$

IV. Ampere's law $\nabla\, x\, H = J + \dfrac{\partial D}{\partial t}$

$$B = \mu_0 (H + M) \quad \textit{General case}$$

$$B = \mu_0 H \quad \textit{Free space}$$

$$B = \mu H \quad \textit{Isotropic linear magnetic medium}$$

Gauss's Law **(Magnetic fields)**	**Integral form:** $\underbrace{\mu_o \oint \vec{H} \cdot d\vec{S}}_{Left} = \underbrace{0}_{Right}$	**Left side:** The number of magnetic field lines – perpendicularly passing through a closed surface. **Right side:** Identically zero.	The total magnetic flux passing through any closed surface is zero. Flux enter the closed surface is same with the flux come out from the surface. The divergence of the magnetic field at any point is zero.
	Differential form: $\underbrace{\mu_o \vec{\nabla} \cdot \vec{H}}_{Left} = \underbrace{0}_{Right}$	**Left side:** Divergence of the magnetic field – the tendency of the field to "flow" away from a point than toward it. **Right side:** Identically zero.	

Faraday's Law	**Integral form:** $\underbrace{\oint_C \vec{E} \cdot d\vec{l}}_{Left} = \underbrace{-\mu_o \int_S \frac{\partial \vec{H}}{\partial t} \cdot d\vec{S}}_{Right}$	**Left side:** The circulation of the vector electric field, $\vec{E}$ around a closed path, C. **Right side:** The rate of change with time (d/dt) of magnetic field, through any surface, $\vec{S}$.	Changing magnetic flux through a surface induces an emf in any boundary path, C of that surface, and a changing magnetic field, $\vec{H}$ induces a circulating electric field.
	Differential form: $\underbrace{\vec{\nabla} \times \vec{E}}_{Left} = \underbrace{-\mu_o \frac{\partial \vec{H}}{\partial t}}_{Right}$	**Left side:** Curl of the electric field, – the tendency of the field lines to circulate around a point. **Right side:** The rate of change of the magnetic field, $\vec{H}$ over time (d/dt)	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time.

Ampere's Law	**Integral form:** $$\underbrace{\oint_C \vec{H}\cdot d\vec{l}}_{Left} = \underbrace{\int_S \left(\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S}}_{Right}$$	**Left side:** The circulation of the magnetic field, $\vec{H}$around a closed path, C. **Right side:** Two sources for the magnetic field, H; a steady conduction current, $\vec{J}_c$ and a changing electric field, $\vec{E}$ through any surface, bounded by closed path, C.	An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path, C that bounds that surface.
	Differential form: $$\underbrace{\vec{\nabla}\times\vec{H}}_{Left} = \underbrace{\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}}_{Right}$$	**Left side:** Curl of the magnetic field, – the tendency of the field lines to circulate around a point. **Right side:** Two terms represent the electric current density, $\vec{J}_c$ and the time rate of change of the electric field, $\vec{E}$.	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time. An electric current, or a changing electric field, through a surface produces a circulating magnetic field, $\vec{H}$around any path that bounds that surface.

Inconsistency of amperes law:

Ampere's circuit law states that the line integral of tangential component of H around a closed path is same as the net current Ienc enclosed by the path.

i.e.

$$\int H.dl = I_{enc}$$

By applying stoke's theorem,

$$\int H.dl \; becomes \int_S J.ds$$

$\therefore$ Therefore, $\Delta \times H = J$ ______ (3.14)

This is true in case of static EM fields.

But in case of time-varying fields, the above Ampere's law shows same inconsistency.

The inconsistency of ampere law for time varying fields is shown in two cases:

1. For static EM fields, we have

$$\Delta \times H = J$$

Applying divergence on both sides, we get,

$$\Delta.(\Delta \times H) = \Delta.J$$

But divergence of curl of a vector field is always zero.

Therefore,

$$\Delta.(\Delta \times H) = 0 = \Delta.J$$

The continuity of current equation is given by

$$\Delta.J = \frac{-dp_v}{dt}$$

Where J = Current density

e_v = Charge density

For static fields, no current is produced, therefore, $e_v = 0 \Rightarrow \Delta.J = 0$

Implies eq. 3.15 is satisfied but for time varying fields, current is produced and therefore,

$$\Delta..J = \frac{-de_v}{dt} \# 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3.16)$$

Eq. (3.15) and eq. (3.16) are contradicting each other.

This is an inconsistency of ampere's law and the Ampere's law must be modified for time varying fields.

2. Consider the typical example of where the surface passes between the capacitor plates.

The figure is shown below.

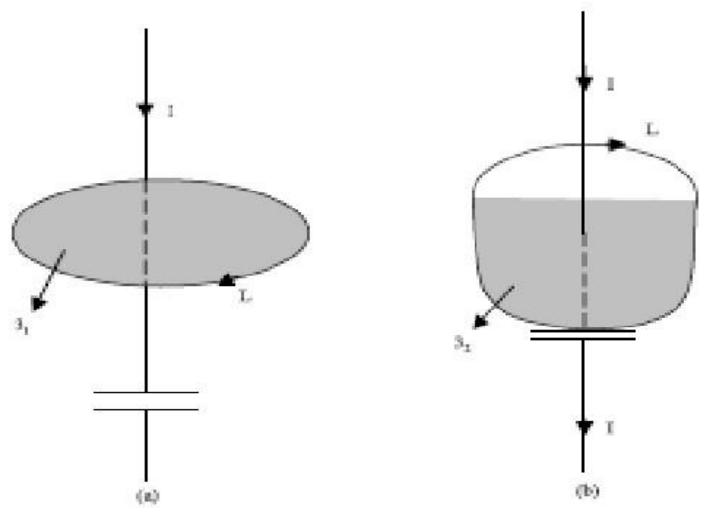


Fig 3.3 (a): Two surfaces of integration which explain the inconsistency of Ampere's law

In fig 3.3(a),

Based on Ampere's circuit law we get figure

$$\oint_L H.dl = \int_{S_1} J.ds = I_{enc} = I \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3.17)$$

In fig 3.3(b), based the ampere's circuit law, we get,

$$\oint_L H.dl = \int_{3_2} J.ds = I_{enc} = 0 \_\_\_\_\_\_\_\_\_\_\_\_ (3.18)$$

Because no conduction current flows through 3_2

i.e. J=0

in both (a) and (b), same closed path is used, but equations 3.17 and 3.18 are different.

This is an inconsistency of Ampere's circuit law.

This inconsistency of Ampere's circuit law in both cases (1) and (2) can be resolved by including displacement current in Ampere's circuit law.

Substituting in (3.19), we get,

$$\Delta \times H = J + \frac{dD}{dt} \_\_\_\_\_\_\_\_\_\_ (3.21)$$

This is the Maxwell equation (based on ampere's circuit Law) for tiem varying fields.

In equation (3.21),

J_d = Displacement current density

J = Conduction current density,

The conduction current density J involves flow of charges. The displacement current density J_d does not involve flow of charges.
Displacement current,

$$I_d = \int Jd.ds = \int \frac{do}{dt}.ds \_\_\_\_\_\_\_\_\_\_ (3.22)$$

Solved problems:

Problem1:

(a) In a cylindrical conductor to the region $0.01 \le r \le 0.02$, $0 < z < 1$ m and the current density is given by,

$$\vec{J} = 10e^{-100r}\hat{a}_\phi \text{ A/m}^2$$

Find the total current crossing the extential of this region with φ = constant plane.

(b) Find the total current in a circular conductor of 4 mm radius if the current density varies according to $J = \frac{10^4}{r}$ A/m^2.

Solution

(a) Total current in the wire is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{r=0.01}^{0.02}\int_{z=0}^{1}\left[10e^{-100r}\hat{a}_\phi\right].\left[rdrdz\hat{a}_\phi\right]$$

$$= \int_{r=0.01}^{0.02}\int_{z=0}^{1} 10re^{-100r}drdz$$

$$= 10\int_{r=0.01}^{0.02} re^{-100r}dr$$

$$I = 10\left[\frac{re^{-100r}}{-100}\Bigg|_{0.01}^{0.02} - \int_{r=0.01}^{0.02}\frac{e^{-100r}}{-100}dr\right]$$

$$= 10\left[-\frac{1}{100}(0.02e^{-2} - 0.01e^{-1}) + \frac{e^{-100r}}{-100\times 100}\Bigg|_{0.01}^{0.02}\right]$$

$$= 2\times 10^{-3}e^{-1}$$

$$= 310^{-3}e^{-2}$$

(b) Total current is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{\phi=0}^{2\pi}\int_{r=0}^{0.004}\frac{10^4}{r}rdrd\phi = 2\pi\times 10^4\int_{r=0}^{0.004}dr = 2\pi\times 10^4\times 0.004 = 80\pi \text{ A}$$

Problem2:

If $\vec{J} = \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)$ A/m^2, calculate the current passing through

(a) A hemispherical shell of 20 cm radius

(b) A spherical shell of 10 cm radius

Solution

Total current is given as $I = \int \vec{J}.d\vec{S}$

Here, $d\vec{S} = r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r$

(a) Total current passing through a hemispherical shell of 20 cm radius is,

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta).(r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r)\Big|_{r=0.2}$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}2\cos\theta r^2\sin\theta\,d\phi\,d\theta\Big|_{r=0.2}$$

$$= 2\pi\times\frac{2}{r}\int_{\theta=0}^{\pi/2}\sin\theta d(\sin\theta)\Big|_{r=0.2}$$

$$= \frac{4\pi}{0.2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 10\pi = 31.42\text{ A}$$

(b) Total current passing through a spherical shell of 10 cm radius is,

$$I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta).(r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r)\Big|_{r=0.1}$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^3}2\cos\theta r^2\sin\theta\,d\phi\,d\theta\Big|_{r=0.1}$$

$$= 2\pi\times\frac{2}{r}\int_{\theta=0}^{\pi}\sin\theta d(\sin\theta)\Big|_{r=0.1}$$

$$= \frac{4\pi}{0.1}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi}$$

$$= 0$$

Problem3:

For the current density, $\vec{J} = 10z\sin^2\phi\, \hat{a}_r$ A/m^2, find the current through the cylindrical surface of $r = 2$, $1 \le z \le 5$ m.

Solution

Total current passing through the cylindrical surface is,

$$I = \int \vec{J}.d\vec{S} = \int_{z=1}^{5}\int_{\phi=0}^{2\pi} (10z\sin^2\phi\, \hat{a}_r).(rd\phi dz\hat{a}_r)\Big|_{r=2} = 10r\left[\frac{z^2}{2}\right]_1^5 \int_{\phi=0}^{2\pi} \sin^2\phi d\phi\Big|_{r=2}$$

$$= 10 \times 2 \times \frac{24}{2} \times \frac{2\pi}{2} = 240\pi = 754 \text{ A}$$

Problem4:

Determine the current density function $\vec{J}$ associated with the magnetic field defined by

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$ A/m (Cartesian)

(b) $\vec{H} = 6r\hat{a}_r + 2r\hat{a}_\phi + 5\hat{a}_z$ A/m (Cylindrical)

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\, \hat{a}_\phi$ A/m (Spherical)

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$

By Ampere's law in Cartesian coordinates,

$$\vec{J} = \nabla \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & 7 & 2x \end{vmatrix} = -2\hat{a}_y \text{ A/m}^2$$

(b) By Ampere's law in cylindrical coordinates,

$$\vec{J} = \nabla \times \vec{H} = \begin{vmatrix} \frac{1}{r}\hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$= \left[\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{a}_\phi + \frac{1}{r}\left[\frac{\partial (rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial \phi}\right]\hat{a}_z$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial \phi}(5) - \frac{\partial}{\partial z}(2r)\right]\hat{a}_r + \left[\frac{\partial}{\partial z}(6r) - \frac{\partial}{\partial r}(5)\right]\hat{a}_\phi + \left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}(r2r) - \frac{\partial}{\partial \phi}(6r)\right]\hat{a}_z$$

$$= \left(\frac{1}{r}\right) \times 4r\hat{a}_z$$

$$= 4\hat{a}_z \text{ A/m}^2$$

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\, \hat{a}_\phi$

By Ampere's law in spherical coordinates,

$$\vec{J} = \nabla \times \vec{H} = \frac{1}{\rho^2 \sin\theta} \begin{vmatrix} \hat{a}_\rho & \hat{a}_\theta & \hat{a}_\phi \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ H_\rho & \rho H_\theta & \rho\sin\theta H_\phi \end{vmatrix}$$

$$= \frac{1}{\rho\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial H_\rho}{\partial\phi} - \frac{\partial}{\partial\rho}(\rho H_\phi)\right]\hat{a}_\theta + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho H_\theta) - \frac{\partial H_\rho}{\partial\theta}\right]\hat{a}_\phi$$

$$= \frac{1}{\rho\sin\theta}\left[\frac{\partial}{\partial\theta}(\cos\theta\sin\theta) - \frac{\partial}{\partial\phi}(3)\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(2\rho) - \frac{\partial}{\partial\rho}(\rho\cos\theta)\right]\hat{a}_\theta + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho 3) - \frac{\partial}{\partial\theta}(2\rho)\right]\hat{a}_\phi$$

$$= \frac{1}{\rho}\left(\frac{\cos 2\theta}{\sin\theta}\right)\hat{a}_\rho - \frac{1}{\rho}\cos\theta\,\hat{a}_\theta + \frac{3}{\rho}\hat{a}_\phi \text{ A/m}^2$$

Problem5:

An infinitely long conductor of radius a is placed such that its axis is along the *z*-axis. The vector magnetic potential, due to a direct current I_0 flowing along $\hat{a}_z$ in the conductor is given by

$$\vec{A} = -\frac{I_0}{4\pi a^2}\mu_0\left(x^2 + y^2\right)\hat{a}_z \text{ Wb/m}$$

Find the corresponding $\vec{H}$. Also confirm the result using Ampere's law.

Solution

The magnetic flux density is given as,

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{I_0}{4\pi a^2}\mu_0(x^2 + y^2) \end{vmatrix} = -\frac{I_0}{2\pi a^2}\mu_0(y\hat{a}_x - x\hat{a}_y)$$

So, the magnetic field intensity is given as,

$$\vec{H} = \frac{\vec{B}}{\mu_0} = -\frac{I_0}{2\pi a^2}(y\hat{a}_x - x\hat{a}_y)$$

We calculate the closed line integral of this field as follows.

$$\oint_L \vec{H}.d\vec{l} = -\frac{I_0}{2\pi a^2}\oint_L (y\hat{a}_x - x\hat{a}_y).(ad\phi\hat{a}_\phi) = -\frac{I_0}{2\pi a^2}\oint_L ad\phi(y\hat{a}_x - x\hat{a}_y).(\hat{a}_\phi)$$

$$= -\frac{I_0}{2\pi a^2}\oint_L ad\phi(y\hat{a}_x - x\hat{a}_y).(-\sin\phi\hat{a}_x + \cos\phi\hat{a}_y)$$

$$= -\frac{I_0}{2\pi a^2}\oint_L ad\phi(-y\sin\phi - x\cos\phi)$$

$$= \frac{I_0}{2\pi a^2}\oint_L ad\phi(a\sin^2\phi + a\cos^2\phi) \qquad \{\because x = r\cos\phi \quad \text{and} \quad y = r\sin\phi\}$$

$$= \frac{I_0}{2\pi}\oint_L d\phi(\sin^2\phi + \cos^2\phi)$$

$$= \frac{I_0}{2\pi}\oint_L d\phi = \frac{I_0}{2\pi}\times 2\pi = I_0$$

Since $\oint_L \vec{H}.d\vec{l} = I_0$, Ampere's law is verified.

Problem6:

Obtain an expression for the self-inductance of a toroid of circular section with '*N*' closely spaced turns.

Solution

Let,

r = Mean radius of the toroid

N = Number of turns

S = Radius of the coil

We have the magnetic field,

$$H = \frac{NI}{2\pi r}$$

total flux linkage per turn is, $\phi = BA = \mu HA = \mu\frac{NI}{2\pi r}\pi S^2 = \frac{\mu NI}{2r}S^2$

Hence, the self-inductance of the toroid is $L = \frac{N\phi}{I} = \frac{\mu N^2 S^2}{2r}$

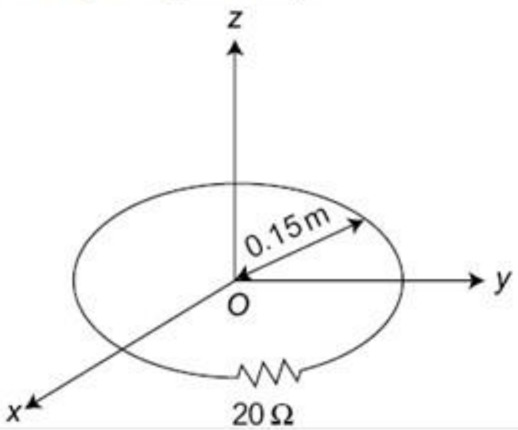
$$\boxed{L = \frac{\mu N^2 S^2}{2r}}$$

Problem7:

The circular loop conductor having a radius of 0.15 m is placed in the *xy* plane. This loop consists of a resistance of 20 Ω as shown in Fig. If the magnetic flux density is

$\vec{B} = 0.5\sin 10^3 t\,\hat{a}_z$ **T**

Find the current flowing through the loop.



Circular loop conductor

Solution

Here since the loop is stationary and the magnetic field is time varying, only the transformer emf is induced.

Transformer emf induced is,

$$V_s = -\iint_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} = -\iint_S \frac{\partial}{\partial t}(0.5\sin 10^3 t\hat{a}_z).(r dr\, d\phi\, \hat{a}_z)$$

$$= -0.5\times 10^3 \cos 10^3 t \int_{r=0}^{0.15}\int_{\phi=0}^{2\pi} r dr\, d\phi$$

$$= -0.5\times 2\pi \times 10^3 \cos 10^3 t\left[\frac{r^2}{2}\right]_0^{0.15}$$

$$= -10^3\pi \cos 10^3 t \times 0.01125$$

$$= -35.34 \cos 10^3 t \text{ V}$$

Problem8:

(a) In free space, $\vec{D} = D_m \sin(\omega t + \beta z)\hat{a}_x$. Using Maxwell's equations, show that

$$\vec{B} = -\frac{\omega\mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y$$

(b) In free space, $\vec{B} = B_m e^{j(\omega t+\beta z)}\hat{a}_y$. Using Maxwell's equations, show that

$$\vec{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t+\beta z)}\hat{a}_x$$

Solution

(a) By Maxwell's equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ and } \vec{D} = \varepsilon_0 \vec{E} \quad \text{or,} \quad \vec{E} = \frac{\vec{D}}{\varepsilon_0} \text{ for free space}$$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{D_m}{\varepsilon_0}\sin(\omega t + \beta z) & 0 & 0 \end{vmatrix} = \frac{D_m}{\varepsilon_0}\frac{\partial}{\partial z}[\sin(\omega t + \beta z)]\hat{a}_y = \frac{D_m \beta}{\varepsilon_0}\cos(\omega t + \beta z)\hat{a}_y$$

$$\vec{B} = -\frac{D_m \beta}{\varepsilon_0}\int \cos(\omega t + \beta z)\hat{a}_y dt = -\frac{D_m \beta}{\omega \varepsilon_0}\sin(\omega t + \beta z)\hat{a}_y$$

or,

Also, for free space,

$$\frac{\omega}{\beta} = v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \Rightarrow \quad \frac{1}{\varepsilon_0} = \mu_0 \left(\frac{\omega}{\beta}\right)^2$$

$$\vec{B} = -\frac{D_m \beta}{\omega \varepsilon_0}\sin(\omega t + \beta z)\hat{a}_y = -\frac{D_m \beta}{\omega} \times \mu_0 \left(\frac{\omega}{\beta}\right)^2 \sin(\omega t + \beta z)\hat{a}_y = -\frac{\omega \mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y$$

$$\boxed{\vec{B} = -\frac{\omega \mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y}$$

(b) By Maxwell's equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} B_m e^{j(\omega t+\beta z)} \hat{a}_y$$

or, $\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -B_m j\omega e^{j(\omega t+\beta z)} \hat{a}_y$

Comparing both sides, we get,

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y = -B_m j\omega e^{j(\omega t+\beta z)} \hat{a}_y$$

$\frac{\partial E_x}{\partial z} = -B_m j\omega e^{j(\omega t+\beta z)}$ $\quad$ $(\because E_z \text{ is not a function of } x)$

$$E_x = \int -B_m j\omega e^{j(\omega t+\beta z)} dz = -B_m j\omega \frac{1}{j\beta} e^{j(\omega t+\beta z)} = -\frac{B_m \omega}{\beta} e^{j(\omega t+\beta z)}$$

$$\boxed{\vec{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t+\beta z)} \hat{a}_x}$$

UNIT – IV
EM WAVE CHARACTERISTICS-I

- Wave Equations for Conducting and Perfect Dielectric Media
- Uniform Plane Waves - Definition, Relation between E & H
- Wave Propagation in Lossless and Conducting Media
- Wave Propagation in Good Conductors and Good Dielectrics
- Illustrative Problems.

Wave equations:

The Maxwell's equations in the differential form are

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D} = \vec{\rho}$$

$$\nabla\cdot\vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (5.29(a))$$

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad (5.29(b))$$

$$\nabla\cdot\vec{E} = 0 \qquad (5.29(c))$$

$$\nabla\cdot\vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla\times\nabla\times\vec{A} = \nabla\cdot\left(\nabla\cdot\vec{A}\right) - \nabla^2 A$$

We can write from 2

$$\nabla\times\nabla\times\vec{E} = \nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E}$$

$$= -\nabla\times\left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Substituting $\nabla\times\vec{H}$ from 1

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free($\nabla\cdot\vec{E} = 0$) medium (eq3)

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

In the same manner for equation eqn 1

$$\nabla\times\nabla\times\vec{H} = \nabla\cdot\left(\nabla\cdot\vec{H}\right) - \nabla^2\vec{H}$$

$$= \sigma\left(\nabla\times\vec{E}\right) + \varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H} = 0$ from eqn 4, we can write

$$\nabla^2\vec{H} = \mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

These two equations

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma \left(\frac{\partial \vec{H}}{\partial t} \right) + \mu\varepsilon \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right)$$

are known as wave equations.

Uniform plane waves:

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wave front or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios

Let us consider a plane wave which has only E_x component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z

i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0$$

The solution to this equation can be written as

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$E_0^+ \ \& E_0^-$ are the amplitude constants (can be determined from boundary conditions).

In the time domain, $\varepsilon_X(z,t) = \mathrm{Re}(E_x(z) e^{jwt})$

$$\varepsilon_X(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

assuming $E_0^+ \ \& E_0^-$ are real constants.

Here, $\varepsilon_X^+(z,t) = E_0^+ \cos(\omega t - \beta z)$ represents the forward traveling wave. The plot of $\varepsilon_X^+(z,t)$ for several values of t is shown in the Figure below

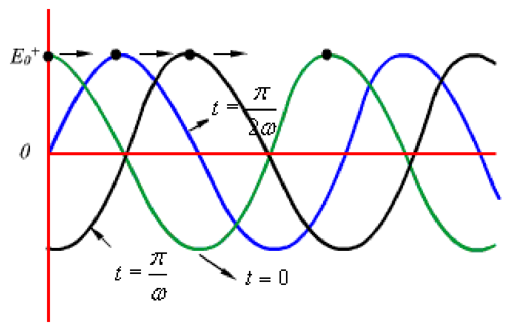


Figure : Plane wave traveling in the + *z* direction

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shown by the dot) i.e. , $\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$

Or, $\omega \Delta t = k \Delta z$

Or, $\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$

When $\Delta t \to 0$,

we write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ = phase velocity v_p .

$\therefore v_p = \frac{\omega}{k}$

If the medium in which the wave is propagating is free space i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

Then $v_p = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points).

i.e., $\left(\omega t - kz\right) - \left[\omega t - k\left(z + \lambda\right)\right] = 2\pi$

or, $k\lambda = 2\pi$

or, $\lambda = \frac{2\pi}{k}$

Substituting $k = \frac{\omega}{v_p}$, $\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$

or, $\lambda f = v_p$

Thus wavelength λ also represents the distance covered in one oscillation of the wave.
Similarly, $\varepsilon^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^+ e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^+ e^{-jkz}\hat{a}_y$$

$$= \frac{E_0^+}{\eta}e^{-jkz}\hat{a}_y = H_0^+ e^{-jkz}\hat{a}_y$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium.

When the wave travels in free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$$

is the intrinsic impedance of the free space.

In the time domain,

$$\vec{H}^+(z,t) = \hat{a}_y\frac{E_0^+}{\eta}\cos(\omega t - \beta z)$$

Which represents the magnetic field of the wave traveling in the +z direction.

For the negative traveling wave,

$$\vec{H}^-(z,t) = -a_y\frac{E_0^+}{\eta}\cos(\omega t + \beta z)$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig below

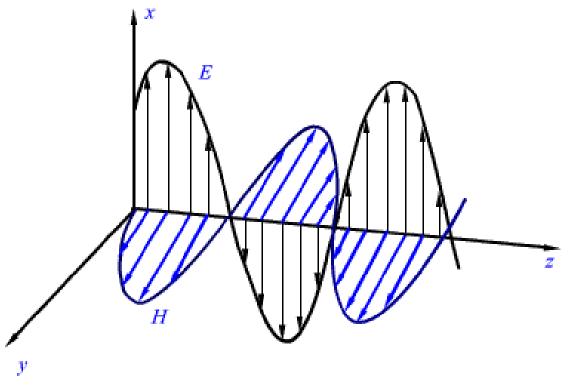


Figure : E & H fields of a particular plane wave at time t.

Solved Problems:

1 The vector amplitude of an electric field associated with a plane wave that propagates in the negative **z** direction in free space is given by $\hat{E}_m = 2a_x + 3a_y \, V/m$

Find the magnetic field strength.

Solution:

The direction of propagation $\mathbf{n}_\beta$ is $-\mathbf{a}_z$. The vector amplitude of the magnetic field is then given by

$$\hat{H}_m = \frac{n_\beta \wedge \hat{E}}{\eta} = \frac{1}{\eta_\circ}\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & -1 \\ 2 & 3 & 0 \end{vmatrix} = \left(\frac{1}{377} 3a_x - 2a_y\right) A/m$$

***note** $\eta_\circ = \sqrt{\frac{\mu_\circ}{\varepsilon_\circ}}$ 120π~377Ω (Appendix D – Table D.1)

2 The phasor electric field expression in a phase is given by

$$\hat{E} = \left[a_x + \hat{E}_y \, a_y + (2 + j5)\, a_z\right] e^{-j2.3(-0.6x + 0.8y)}$$

Find the following:

1. $\hat{E}_y$.
2. Vector magnetic field, assuming $\mu = \mu_\circ$ and $\varepsilon = \varepsilon_\circ$.
3. Frequency and wavelength of this wave.

Solution:

1. The general expression for a uniform plane wave propagating in an arbitrary direction is given by

$$\hat{E} = \hat{E}_m e^{-j\beta \cdot r}$$

where the amplitude vector $\hat{E}_m$, in general, has components in the x, y, and z directions. Comparing equation 6.3 with the general field equation for the plane wave propagating in an arbitrary direction, we obtain

$$\begin{aligned}\beta \cdot r &= \beta_x x + \beta_y y + \beta_z z \\ &= \beta (\cos\theta_x x + \cos\theta_y y + \cos\theta_z z) \\ &= 2.3(-0.6x + 0.8y + 0)\end{aligned}$$

Hence, a unit vector in the direction of propagation $\mathbf{n}_\beta$ is given by

$$\mathbf{n}_\beta = -0.6a_x + 0.8a_y.$$

Because the electric field $\hat{E}$ must be perpendicular to the direction of propagation $\mathbf{n}_\beta$, it must satisfy the following relations:

$$\mathbf{n}_\beta \cdot \hat{E} = 0$$

Therefore, $(-0.6a_x + 0.8a_y) \cdot \left[a_x + \hat{E}_y a_y + (2 + j5) a_z\right] = 0$

Or

$$-0.6 + 0.8\hat{E}_y = 0$$

Hence, $\hat{E}_y$ = 0.75. The electric field is given by

$$\hat{E} = \left[a_x + \hat{E}_y a_y + (2 + j5) a_z\right] e^{-j2.3(-0.6x+0.8y)}$$

2. The vector magnetic field $\hat{H}$ is given by

$$\hat{H} = \frac{1}{\eta} n_\beta \wedge \hat{E} = \frac{1}{377}\begin{vmatrix} a_x & a_y & a_z \\ -0.6 & 0.8 & 0 \\ 1 & 0.75 & 2 + j5 \end{vmatrix}$$

so that

$$\hat{H}_x = \frac{0.8(2 + j5)}{377} = (4.24 + j10.6)*10^{-3}$$

$$\hat{H}_y = \frac{0.6(2+j5)}{377} = (3.18 - j7.95)*10^{-3}$$

$$\hat{H}_z = \frac{0.6(0.75)+0.8}{377} = -3.31*10^{-3}$$

The vector magnetic field is then given by

$$\hat{H} = \left(\hat{H}_x\, a_x + \hat{H}_y\, a_y + \hat{H}_z\, a_z\right) e^{-j2.3(-0.6x+0.8y)}$$

3. The wavelength **λ** is given by

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2.3} = 2.73\ m$$

and the frequency

$$f = \frac{c}{\lambda} = \frac{3*10^8}{2.73} = 0.11 GHz$$

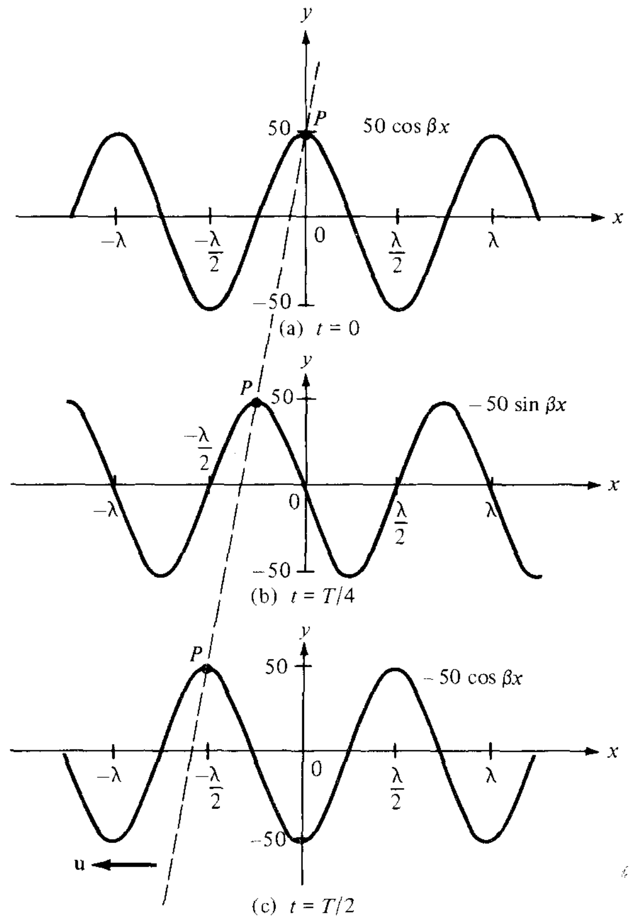


Figure 10.3 For Example 10.1; wave travels along $-\mathbf{a}_x$.

PRACTICE EXERCISE 10.1

In free space, $\mathbf{H} = 0.1 \cos (2 \times 10^8 t - kx)\, \mathbf{a}_y$ A/m. Calculate

(a) k, λ, and T

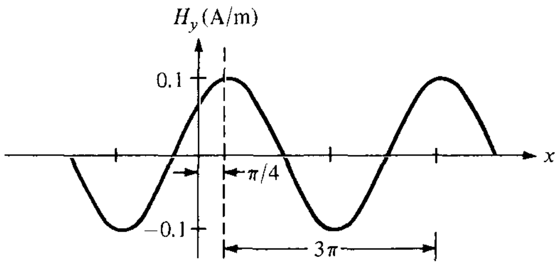
(b) The time t_1 it takes the wave to travel $\lambda/8$

(c) Sketch the wave at time t_1.

Answer: (a) 0.667 rad/m, 9.425 m, 31.42 ns, (b) 3.927 ns, (c) see Figure 10.4.

WAVE PROPAGATION IN LOSSY DIELECTRICS

As mentioned in Section 10.1, wave propagation in lossy dielectrics is a general case from which wave propagation in other types of media can be derived as special cases. Therefore, this section is foundational to the next three sections.



A **lossy dielectric** is a medium in which an EM wave loses power as it propagates due to poor conduction.

In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations (see Table 9.2) become

$$\nabla \cdot \mathbf{E}_s = 0 \tag{10.11}$$

$$\nabla \cdot \mathbf{H}_s = 0 \tag{10.12}$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \tag{10.13}$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s \tag{10.14}$$

Taking the curl of both sides of eq. (10.13) gives

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu \; \nabla \times \mathbf{H}_s \tag{10.15}$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{10.16}$$

to the left-hand side of eq. (10.15) and invoking eqs. (10.11) and (10.14), we obtain

$$\nabla(\underbrace{\cancel{\nabla \cdot \mathbf{E}_s}}_{0}) - \nabla^2\mathbf{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}_s$$

or

$$\boxed{\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0} \tag{10.17}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \tag{10.18}$$

and γ is called the *propagation constant* (in per meter) of the medium. By a similar procedure, it can be shown that for the **H** field,

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0 \tag{10.19}$$

Equations (10.17) and (10.19) are known as homogeneous vector *Helmholtz's equations* or simply vector *wave equations.* In Cartesian coordinates, eq. (10.17), for example, is equivalent to three scalar wave equations, one for each component of **E** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

Since γ in eqs. (10.17) to (10.19) is a complex quantity, we may let

$$\boxed{\gamma = \alpha + j\beta} \tag{10.20}$$

We obtain α and β from eqs. (10.18) and (10.20) by noting that

$$-\text{Re}\, \gamma^2 = \beta^2 - \alpha^2 = \omega^2 \mu\varepsilon \tag{10.21}$$

and

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu \sqrt{\sigma^2 + \omega^2\varepsilon^2} \tag{10.22}$$

From eqs. (10.21) and (10.22), we obtain

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \tag{10.23}$$

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]} \tag{10.24}$$

Without loss of generality, if we assume that the wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x-component, then

$$\mathbf{E}_s = E_{xs}(z)\mathbf{a}_x \tag{10.25}$$

Substituting this into eq. (10.17) yields

$$(\nabla^2 - \gamma^2)E_{xs}(z) \tag{10.26}$$

Hence

$$\underbrace{\cancel{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}}_{0} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

or

$$\left[\frac{d^2}{dz^2} - \gamma^2\right] E_{xs}(z) = 0 \tag{10.27}$$

This is a scalar wave equation, a linear homogeneous differential equation, with solution (see Case 2 in Example 6.5)

$$E_{xs}(z) = E_o e^{-\gamma z} + E_o' e^{\gamma z} \tag{10.28}$$

where E_o and E_o' are constants. The fact that the field must be finite at infinity requires that $E_o' = 0$. Alternatively, because $e^{\gamma z}$ denotes a wave traveling along $-\mathbf{a}_z$ whereas we assume wave propagation along $\mathbf{a}_z$, $E_o' = 0$. Whichever way we look at it, $E_o' = 0$. Inserting the time factor $e^{j\omega t}$ into eq. (10.28) and using eq. (10.20), we obtain

$$\mathbf{E}(z, t) = \text{Re}\,[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}\,(E_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_x)$$

or

$$\boxed{\mathbf{E}(z, t) = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x} \tag{10.29}$$

A sketch of $|\mathbf{E}|$ at times $t = 0$ and $t = \Delta t$ is portrayed in Figure 10.5, where it is evident that $\mathbf{E}$ has only an x-component and it is traveling along the $+z$-direction. Having obtained $\mathbf{E}(z, t)$, we obtain $\mathbf{H}(z, t)$ either by taking similar steps to solve eq. (10.19) or by using eq. (10.29) in conjunction with Maxwell's equations as we did in Example 9.8. We will eventually arrive at

$$\mathbf{H}(z, t) = \text{Re}\,(H_o e^{-\alpha z} e^{j(\omega t - \beta z)}\,\mathbf{a}_y) \tag{10.30}$$

where

$$H_o = \frac{E_o}{\eta} \tag{10.31}$$

and η is a complex quantity known as the *intrinsic impedance* (in ohms) of the medium. It can be shown by following the steps taken in Example 9.8 that

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\underline{/\theta_\eta} = |\eta|e^{j\theta_\eta} \tag{10.32}$$

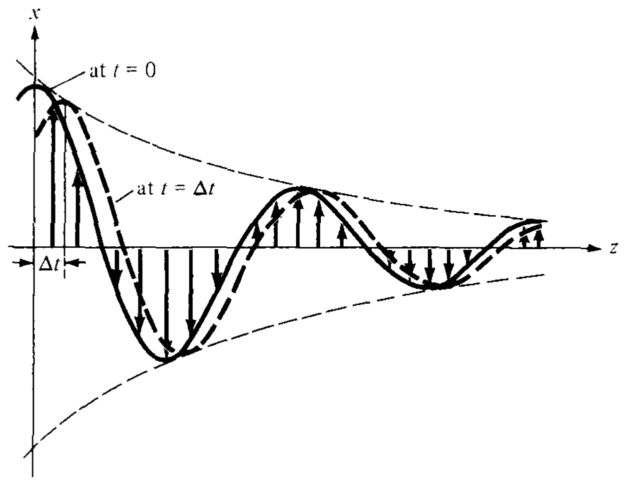


Figure 10.5 E-field with x-component traveling along $+z$-direction at times $t = 0$ and $t = \Delta t$; arrows indicate instantaneous values of E.

with

$$\boxed{|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}} \tag{10.33}$$

where $0 \leq \theta_\eta \leq 45°$. Substituting eqs. (10.31) and (10.32) into eq. (10.30) gives

$$\mathbf{H} = \text{Re}\left[\frac{E_o}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_y\right]$$

or

$$\boxed{\mathbf{H} = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_y} \tag{10.34}$$

Notice from eqs. (10.29) and (10.34) that as the wave propagates along $\mathbf{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and hence α is known as the *attenuation constant* or *attenuation factor* of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to e^{-1} of the original value whereas an increase of 1 neper indicates an increase by a factor of e. Hence, for voltages

$$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB} \tag{10.35}$$

From eq. (10.23), we notice that if $\sigma = 0$, as is the case for a lossless medium and free space, $\alpha = 0$ and the wave is not attenuated as it propagates. The quantity β is a measure of the phase shift per length and is called the *phase constant* or *wave number.* In terms of β, the wave velocity u and wavelength λ are, respectively, given by [see eqs. (10.7b) and (10.8)]

$$u = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.36}$$

We also notice from eqs. (10.29) and (10.34) that $\mathbf{E}$ and $\mathbf{H}$ are out of phase by θ_η at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, $\mathbf{E}$ leads $\mathbf{H}$ (or $\mathbf{H}$ lags $\mathbf{E}$) by θ_η. Finally, we notice that the ratio of the magnitude of the conduction current density $\mathbf{J}$ to that of the displacement current density $\mathbf{J}_d$ in a lossy medium is

$$\frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega\varepsilon \mathbf{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

or

$$\boxed{\tan\theta = \frac{\sigma}{\omega\varepsilon}} \tag{10.37}$$

where $\tan \theta$ is known as the *loss tangent* and θ is the *loss angle* of the medium as illustrated in Figure 10.6. Although a line of demarcation between good conductors and lossy dielectrics is not easy to make, $\tan \theta$ or θ may be used to determine how lossy a medium is. A medium is said to be a good (lossless or perfect) dielectric if $\tan \theta$ is very small ($\sigma \ll \omega\varepsilon$) or a good conductor if $\tan \theta$ is very large ($\sigma \gg \omega\varepsilon$). From the viewpoint of wave propagation, the characteristic behavior of a medium depends not only on its constitutive parameters σ, ε, and μ but also on the frequency of operation. A medium that is regarded as a good conductor at low frequencies may be a good dielectric at high frequencies. Note from eqs. (10.33) and (10.37) that

$$\theta = 2\theta_\eta \tag{10.38}$$

From eq. (10.14)

$$\begin{aligned} \nabla \times \mathbf{H}_s &= (\sigma + j\omega\varepsilon)\mathbf{E}_s = j\omega\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]\mathbf{E}_s \\ &= j\omega\varepsilon_c\mathbf{E}_s \end{aligned} \tag{10.39}$$

where

$$\boxed{\varepsilon_c = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]} \tag{10.40a}$$

or

$$\varepsilon_c = \varepsilon' - j\varepsilon'' \tag{10.40b}$$

and $\varepsilon' = \varepsilon$, $\varepsilon'' = \sigma/\omega$; ε_c is called the *complex permittivity* of the medium. We observe that the ratio of ε'' to ε' is the loss tangent of the medium; that is,

$$\tan \theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} \tag{10.41}$$

In subsequent sections, we will consider wave propagation in other types of media, which may be regarded as special cases of what we have considered here. Thus we will simply deduce the governing formulas from those obtained for the general case treated in this section. The student is advised not just to memorize the formulas but to observe how they are easily obtained from the formulas for the general case.

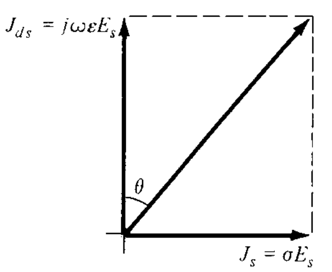


Figure 10.6 Loss angle of a lossy medium.

PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\varepsilon$. It is a special case of that in Section 10.3 except that

$$\boxed{\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o\varepsilon_r, \qquad \mu = \mu_o\mu_r} \tag{10.42}$$

Substituting these into eqs. (10.23) and (10.24) gives

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon} \tag{10.43a}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.43b}$$

Also

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\,\underline{/0^\circ} \tag{10.44}$$

and thus **E** and **H** are in time phase with each other.

PLANE WAVES IN FREE SPACE

This is a special case of what we considered in Section 10.3. In this case,

$$\boxed{\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o} \tag{10.45}$$

This may also be regarded as a special case of Section 10.4. Thus we simply replace ε by ε_o and μ by μ_o in eq. (10.43) or we substitute eq. (10.45) directly into eqs. (10.23) and (10.24). Either way, we obtain

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c} \tag{10.46a}$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.46b}$$

where $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters in eq. (10.45) into eq. (10.33), $\theta_\eta = 0$ and $\eta = \eta_o$, where η_o is called the *intrinsic impedance of free space* and is given by

$$\boxed{\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\ \Omega} \tag{10.47}$$

$$\mathbf{E} = E_o \cos(\omega t - \beta z)\, \mathbf{a}_x \tag{10.48a}$$

then

$$\mathbf{H} = H_o \cos(\omega t - \beta z)\, \mathbf{a}_y = \frac{E_o}{\eta_o} \cos(\omega t - \beta z)\, \mathbf{a}_y \tag{10.48b}$$

The plots of **E** and **H** are shown in Figure 10.7(a). In general, if $\mathbf{a}_E$, $\mathbf{a}_H$, and $\mathbf{a}_k$ are unit vectors along the **E** field, the **H** field, and the direction of wave propagation; it can be shown that (see Problem 10.14)

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$$

or

$$\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

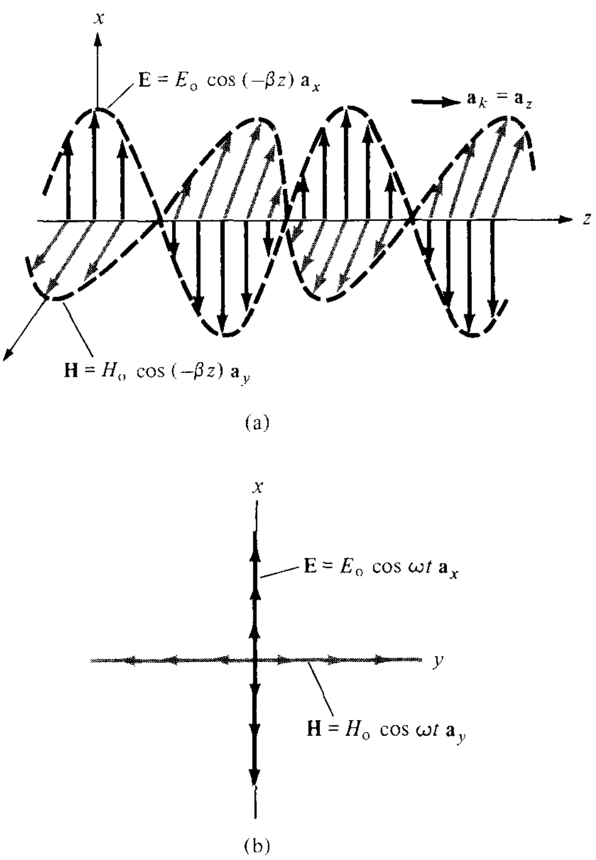


Figure 10.7 **(a)** Plot of **E** and **H** as functions of z at $t = 0$; **(b)** plot of **E** and **H** at $z = 0$. The arrows indicate instantaneous values.

or

$$\boxed{\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k} \tag{10.49}$$

Both **E** and **H** fields (or EM waves) are everywhere normal to the direction of wave propagation, $\mathbf{a}_k$. That means, the fields lie in a plane that is transverse or orthogonal to the direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called a *transverse electromagnetic* (TEM) wave. Each of **E** and **H** is called a *uniform plane wave* because **E** (or **H**) has the same magnitude throughout any transverse plane, defined by z = constant. The direction in which the electric field points is the *polarization* of a TEM wave.[3] The wave in eq. (10.29), for example, is polarized in the x-direction. This should be observed in Figure 10.7(b), where an illustration of uniform plane waves is given. A uniform plane wave cannot exist physically because it stretches to infinity and would represent an infinite energy. However, such waves are characteristically simple but fundamentally important. They serve as approximations to practical waves, such as from a radio antenna, at distances sufficiently far from radiating sources. Although our discussion after eq. (10.48) deals with free space, it also applies for any other isotropic medium.

PLANE WAVES IN GOOD CONDUCTORS

This is another special case of that considered in Section 10.3. A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$ so that $\sigma/\omega\varepsilon \to \infty$; that is,

$$\boxed{\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r} \tag{10.50}$$

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \tag{10.51a}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.51b}$$

Also,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\,\underline{/45^\circ} \tag{10.52}$$

and thus **E** leads **H** by 45°. If

$$\mathbf{E} = E_o e^{-\alpha z}\cos(\omega t - \beta z)\,\mathbf{a}_x \tag{10.53a}$$

then

$$\mathbf{H} = \frac{E_o}{\sqrt{\dfrac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45°)\, \mathbf{a}_y \tag{10.53b}$$

Therefore, as **E** (or **H**) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ, shown in Figure 10.8, through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

or

$$\boxed{\delta = \frac{1}{\alpha}} \tag{10.54a}$$

The **skin depth** is a measure of the depth to which an **EM** wave can penetrate the medium.

Equation (10.54a) is generally valid for any material medium. For good conductors, eqs. (10.51a) and (10.54a) give

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{10.54b}$$

The illustration in Figure 10.8 for a good conductor is exaggerated. However, for a partially conducting medium, the skin depth can be considerably large. Note from eqs. (10.51a), (10.52), and (10.54b) that for a good conductor.

$$\eta = \frac{1}{\sigma\delta}\sqrt{2}\, e^{j\pi/4} = \frac{1+j}{\sigma\delta} \tag{10.55}$$

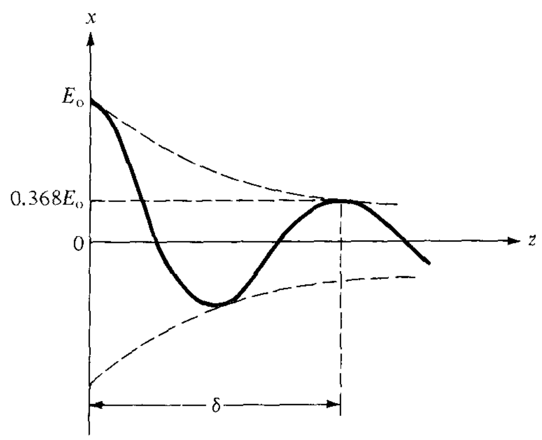


TABLE 10.2 Skin Depth in Copper*

Frequency (Hz)	10	60	100	500	10^4	10^8	10^{10}
Skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

*For copper, $\sigma = 5.8 \times 10^7$ mhos/m, $\mu = \mu_o$, $\delta = 66.1/\sqrt{f}$ (in mm).

Also for good conductors, eq. (10.53a) can be written as

$$\mathbf{E} = E_o e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) \mathbf{a}_x$$

showing that δ measures the exponential damping of the wave as it travels through the conductor. The skin depth in copper at various frequencies is shown in Table 10.2. From the table, we notice that the skin depth decreases with increase in frequency. Thus, **E** and **H** can hardly propagate through good conductors.

The phenomenon whereby field intensity in a conductor rapidly decreases is known as *skin effect.* The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface. For a wire of radius a, for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness δ as shown in Figure 10.9. Skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, and electromagnetic shielding. It is used to advantage in many applications. For example, because the skin depth in silver is very small, the difference in performance between a pure silver component and a silver-plated brass component is negligible, so silver plating is often used to reduce material cost of waveguide components. For the same reason, hollow tubular conductors are used instead of solid conductors in outdoor television antennas. Effective electromagnetic shielding of electrical devices can be provided by conductive enclosures a few skin depths in thickness.

The skin depth is useful in calculating the *ac resistance* due to skin effect. The resistance in eq. (5.16) is called the *dc resistance,* that is,

$$R_{dc} = \frac{\ell}{\sigma S} \tag{5.16}$$

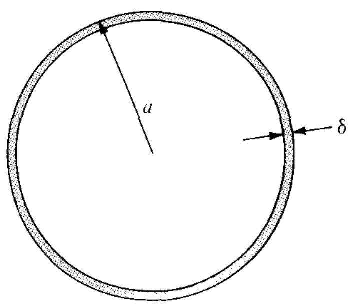


Figure 10.9 Skin depth at high frequencies, $\delta \ll a$.

We define the *surface or skin resistance* R_s (in Ω/m^2) as the real part of the η for a good conductor. Thus from eq. (10.55)

$$\boxed{R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}} \tag{10.56}$$

This is the resistance of a unit width and unit length of the conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross-sectional area $1 \times \delta$. Thus for a given width w and length ℓ, the ac resistance is calculated using the familiar dc resistance relation of eq. (5.16) and assuming a uniform current flow in the conductor of thickness δ, that is,

$$\boxed{R_{ac} = \frac{\ell}{\sigma\delta w} = \frac{R_s \ell}{w}} \tag{10.57}$$

where $S \simeq \delta w$. For a conductor wire of radius a (see Figure 10.9), $w = 2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\dfrac{\ell}{\sigma 2\pi a \delta}}{\dfrac{\ell}{\sigma \pi a^2}} = \frac{a}{2\delta}$$

Since $\delta \ll a$ at high frequencies, this shows that R_{ac} is far greater than R_{dc}. In general, the ratio of the ac to the dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases. Also, although the bulk of the current is nonuniformly distributed over a thickness of 5δ of the conductor, the power loss is the same as though it were uniformly distributed over a thickness of δ and zero elsewhere. This is one more reason why δ is referred to as the skin depth.

EXAMPLE

A lossy dielectric has an intrinsic impedance of $200 \angle 30^\circ$ Ω at a particular frequency. If, at that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$\mathbf{H} = 10\, e^{-\alpha x} \cos\left(\omega t - \frac{1}{2}x\right) \mathbf{a}_y \text{ A/m}$$

find $\mathbf{E}$ and α. Determine the skin depth and wave polarization.

Solution:

The given wave travels along $\mathbf{a}_x$ so that $\mathbf{a}_k = \mathbf{a}_x$; $\mathbf{a}_H = \mathbf{a}_y$, so

$$-\mathbf{a}_E = \mathbf{a}_k \times \mathbf{a}_H = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

or

$$\mathbf{a}_E = -\mathbf{a}_z$$

Also $H_o = 10$, so

$$\frac{E_o}{H_o} = \eta = 200 \underline{/30^\circ} = 200\, e^{j\pi/6} \rightarrow E_o = 2000e^{j\pi/6}$$

Except for the amplitude and phase difference, **E** and **H** always have the same form. Hence

$$\mathbf{E} = \text{Re}\,(2000e^{j\pi/6}e^{-\gamma x}e^{j\omega t}\mathbf{a}_E)$$

or

$$\mathbf{E} = -2e^{-\alpha x}\cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right)\mathbf{a}_z \text{ kV/m}$$

Knowing that $\beta = 1/2$, we need to determine α. Since

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

and

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1}{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1}\right]^{1/2}$$

But $\dfrac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^\circ = \sqrt{3}$. Hence,

$$\frac{\alpha}{\beta} = \left[\frac{2 - 1}{2 + 1}\right]^{1/2} = \frac{1}{\sqrt{3}}$$

or

$$\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887 \text{ Np/m}$$

and

$$\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464 \text{ m}$$

The wave has an E_z component; hence it is polarized along the z-direction.

PRACTICE EXERCISE

A plane wave propagating through a medium with $\varepsilon_r = 8$, $\mu_r = 2$ has $\mathbf{E} = 0.5\, e^{-z/3} \sin(10^8 t - \beta z)\, \mathbf{a}_x$ V/m. Determine

(a) β

(b) The loss tangent

(c) Wave impedance

(d) Wave velocity

(e) **H** field

Answer: (a) 1.374 rad/m, (b) 0.5154, (c) 177.72 $\angle 13.63°$ Ω, (d) 7.278×10^7 m/s, (e) $2.817e^{-z/3} \sin(10^8 t - \beta z - 13.63°)\mathbf{a}_y$ mA/m.

EXAMPLE

In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$, and $\mathbf{H} = -0.1 \cos(\omega t - z)\, \mathbf{a}_x + 0.5 \sin(\omega t - z)\mathbf{a}_y$ A/m, calculate ε_r, ω, and $\mathbf{E}$.

Solution:

In this case, $\sigma = 0$, $\alpha = 0$, and $\beta = 1$, so

$$\eta = \sqrt{\mu/\varepsilon} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}}$$

or

$$\sqrt{\varepsilon_r} = \frac{120\pi}{\eta} = \frac{120\pi}{60\pi} = 2 \quad \rightarrow \quad \varepsilon_r = 4$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\,\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{4} = \frac{2\omega}{c}$$

or

$$\omega = \frac{\beta c}{2} = \frac{1\,(3 \times 10^8)}{2} = 1.5 \times 10^8 \text{ rad/s}$$

From the given **H** field, **E** can be calculated in two ways: using the techniques (based on Maxwell's equations) developed in this chapter or directly using Maxwell's equations as in the last chapter.

Method 1: To use the techniques developed in this chapter, we let

$$\mathbf{E} = \mathbf{H}_1 + \mathbf{H}_2$$

where $\mathbf{H}_1 = -0.1 \cos(\omega t - z)\, \mathbf{a}_x$ and $\mathbf{H}_2 = 0.5 \sin(\omega t - z)\, \mathbf{a}_y$ and the corresponding electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

where $\mathbf{E}_1 = E_{1o} \cos(\omega t - z)\, \mathbf{a}_{E_1}$ and $\mathbf{E}_2 = E_{2o} \sin(\omega t - z)\, \mathbf{a}_{E_2}$. Notice that although $\mathbf{H}$ has components along $\mathbf{a}_x$ and $\mathbf{a}_y$, it has no component along the direction of propagation; it is therefore a TEM wave.

For $\mathbf{E}_1$,

$$\mathbf{a}_{E_1} = -(\mathbf{a}_k \times \mathbf{a}_{H_1}) = -(\mathbf{a}_z \times -\mathbf{a}_x) = \mathbf{a}_y$$

$$E_{1o} = \eta H_{1o} = 60\pi\,(0.1) = 6\pi$$

Hence

$$\mathbf{E}_1 = 6\pi \cos(\omega t - z)\, \mathbf{a}_y$$

For $\mathbf{E}_2$,

$$\mathbf{a}_{E_2} = -(\mathbf{a}_k \times \mathbf{a}_{H_2}) = -(\mathbf{a}_z \times \mathbf{a}_y) = \mathbf{a}_x$$

$$E_{2o} = \eta H_{2o} = 60\pi\,(0.5) = 30\pi$$

Hence

$$\mathbf{E}_2 = 30\pi \sin(\omega t - z)\mathbf{a}_x$$

Adding $\mathbf{E}_1$ and $\mathbf{E}_2$ gives $\mathbf{E}$; that is,

$$\mathbf{E} = 94.25 \sin(1.5 \times 10^8 t - z)\, \mathbf{a}_x + 18.85 \cos(1.5 \times 10^8 t - z)\, \mathbf{a}_y \text{ V/m}$$

Method 2: We may apply Maxwell's equations directly.

$$\nabla \times \mathbf{H} = \underbrace{\cancel{\sigma}\mathbf{E}}_{0} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \quad \mathbf{E} = \frac{1}{\varepsilon} \int \nabla \times H\, dt$$

because $\sigma = 0$. But

$$\nabla \times \mathbf{H} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x(z) & H_y(z) & 0 \end{vmatrix} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x + \frac{\partial H_x}{\partial z}\mathbf{a}_y$$
$$= H_{2o} \cos(\omega t - z)\, \mathbf{a}_x + H_{1o} \sin(\omega t - z)\mathbf{a}_y$$

where $H_{1o} = -0.1$ and $H_{2o} = 0.5$. Hence

$$\mathbf{E} = \frac{1}{\varepsilon} \int \nabla \times \mathbf{H}\, dt = \frac{H_{2o}}{\varepsilon\omega} \sin(\omega t - z)\, \mathbf{a}_x - \frac{H_{1o}}{\varepsilon\omega} \cos(\omega t - z)\, \mathbf{a}_y$$
$$= 94.25 \sin(\omega t - z)\, \mathbf{a}_x + 18.85 \cos(\omega t - z)\, \mathbf{a}_y \text{ V/m}$$

as expected.

PRACTICE EXERCISE

A plane wave in a nonmagnetic medium has $\mathbf{E} = 50 \sin(10^8 t + 2z)\, \mathbf{a}_y$ V/m. Find

(a) The direction of wave propagation

(b) $\lambda, f,$ and ε_r

(c) $\mathbf{H}$

Answer: (a) along $-z$ direction, (b) 3.142 m, 15.92 MHz, 36, (c) 0.7958 $\sin(10^8 t + 2z)\, \mathbf{a}_x$ A/m.

EXAMPLE 1

A uniform plane wave propagating in a medium has

$$\mathbf{E} = 2e^{-\alpha z} \sin(10^8 t - \beta z)\, \mathbf{a}_y \text{ V/m.}$$

If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ mhos/m, find α, β, and $\mathbf{H}$.

Solution:

We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times \dfrac{10^{-9}}{36\pi}} = 3393 \gg 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\begin{aligned} \alpha = \beta &= \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)(3)}{2}\right]^{1/2} \\ &= 61.4 \\ \alpha &= 61.4 \text{ Np/m}, \qquad \beta = 61.4 \text{ rad/m} \end{aligned}$$

Also

$$\begin{aligned} |\eta| &= \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)}{3}\right]^{1/2} \\ &= \sqrt{\frac{800\pi}{3}} \end{aligned}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 3393 \quad \rightarrow \quad \theta_\eta = 45^\circ = \pi/4$$

Hence

$$\mathbf{H} = H_o e^{-\alpha z} \sin\left(\omega t - \beta z - \frac{\pi}{4}\right) \mathbf{a}_H$$

where

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

and

$$H_o = \frac{E_o}{|\eta|} = 2\sqrt{\frac{3}{800\pi}} = 69.1 \times 10^{-3}$$

Thus

$$\mathbf{H} = -69.1\, e^{-61.4z} \sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right) \mathbf{a}_x \text{ mA/m}$$

PRACTICE EXERCISE 10.4

A plane wave traveling in the $+y$-direction in a lossy medium ($\varepsilon_r = 4$, $\mu_r = 1$, $\sigma = 10^{-2}$ mhos/m) has $\mathbf{E} = 30 \cos(10^9 \pi\, t + \pi/4)\, \mathbf{a}_z$ V/m at $y = 0$. Find

(a) $\mathbf{E}$ at $y = 1$ m, $t = 2$ ns

(b) The distance traveled by the wave to have a phase shift of 10°

(c) The distance traveled by the wave to have its amplitude reduced by 40%

(d) $\mathbf{H}$ at $y = 2$ m, $t = 2$ ns

Answer: (a) $2.787\mathbf{a}_z$ V/m, (b) 8.325 mm, (c) 542 mm, (d) $-4.71\mathbf{a}_x$ mA/m

XAMPLE

A plane wave $\mathbf{E} = E_o \cos(\omega t - \beta z)\, \mathbf{a}_x$ is incident on a good conductor at $z = 0$. Find the current density in the conductor.

Solution:

Since the current density $\mathbf{J} = \sigma \mathbf{E}$, we expect $\mathbf{J}$ to satisfy the wave equation in eq. (10.17), that is,

$$\nabla^2 \mathbf{J}_s - \gamma^2 \mathbf{J}_s = 0$$

Also the incident $\mathbf{E}$ has only an x-component and varies with z. Hence $\mathbf{J} = J_x(z, t)\, \mathbf{a}_x$ and

$$\frac{d^2}{dz^2} J_{sx} - \gamma^2 J_{sx} = 0$$

which is an ordinary differential equation with solution (see Case 2 of Example 6.5)

$$J_{sx} = Ae^{-\gamma z} + Be^{+\gamma z}$$

The constant B must be zero because J_{sx} is finite as $z \to \infty$. But in a good conductor, $\sigma \gg \omega\varepsilon$ so that $\alpha = \beta = 1/\delta$. Hence

$$\gamma = \alpha + j\beta = \alpha(1 + j) = \frac{(1 + j)}{\delta}$$

and

$$J_{sx} = Ae^{-z(1+j)/\delta}$$

or

$$J_{sx} = J_{sx}(0)\, e^{-z(1+j)/\delta}$$

where J_{sx} (0) is the current density on the conductor surface.

PRACTICE EXERCISE

Due to the current density of Example 10.5, find the magnitude of the total current through a strip of the conductor of infinite depth along z and width w along y.

Answer: $\dfrac{J_{sx}(0)w\delta}{\sqrt{2}}$

EXAMPLE

For the copper coaxial cable of Figure 7.12, let $a = 2$ mm, $b = 6$ mm, and $t = 1$ mm. Calculate the resistance of 2 m length of the cable at dc and at 100 MHz.

Solution:
Let

$$R = R_o + R_i$$

where R_o and R_i are the resistances of the inner and outer conductors.
At dc,

$$R_i = \frac{\ell}{\sigma S} = \frac{\ell}{\sigma\pi a^2} = \frac{2}{5.8 \times 10^7 \pi [2 \times 10^{-3}]^2} = 2.744 \text{ m}\Omega$$

$$\begin{aligned} R_o &= \frac{\ell}{\sigma S} = \frac{\ell}{\sigma\pi[[b + t]^2 - b^2]} = \frac{\ell}{\sigma\pi[t^2 + 2bt]} \\ &= \frac{2}{5.8 \times 10^7 \pi\, [1 + 12] \times 10^{-6}} \\ &= 0.8429 \text{ m}\Omega \end{aligned}$$

Hence $R_{dc} = 2.744 + 0.8429 = 3.587$ mΩ

At $f = 100$ MHz,

$$R_i = \frac{R_s \ell}{w} = \frac{\ell}{\sigma \delta 2\pi a} = \frac{\ell}{2\pi a}\sqrt{\frac{\pi f \mu}{\sigma}}$$
$$= \frac{2}{2\pi \times 2 \times 10^{-3}}\sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$
$$= 0.41\ \Omega$$

Since $\delta = 6.6\ \mu\text{m} \ll t = 1$ mm, $w = 2\pi b$ for the outer conductor. Hence,

$$R_o = \frac{R_s \ell}{w} = \frac{\ell}{2\pi b}\sqrt{\frac{\pi f \mu}{\sigma}}$$
$$= \frac{2}{2\pi \times 6 \times 10^{-3}}\sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$
$$= 0.1384\ \Omega$$

Hence,

$$R_{ac} = 0.41 + 0.1384 = 0.5484\ \Omega$$

which is about 150 times greater than R_{dc}. Thus, for the same effective current i, the ohmic loss (i^2R) of the cable at 100 MHz is far greater than the dc power loss by a factor of 150.

PRACTICE EXERCISE

For an aluminum wire having a diameter 2.6 mm, calculate the ratio of ac to dc resistance at

(a) 10 MHz

(b) 2 GHz

Answer: (a) 24.16, (b) 341.7.

POWER AND THE POYNTING VECTOR

As mentioned before, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{10.58a}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{10.58b}$$

UNIT - V
EM WAVE CHARACTERISTICS – II

- Reflection and Refraction of Plane Waves –
  - Normal incidence for both perfect Conductors
  - perfect Dielectrics
- Brewster Angle
- Critical Angle
- Total Internal Reflection S
- Surface Impedance,
- Poynting Vector and
- Poynting Theorem – Applications
- Illustrative Problems.

Plane Wave Propagation at Arbitrary Angle

Plane waves are not normally incident, so now we must consider the general problem of a plane wave propagating along a specified axis that is arbitrarily relative to a rectangular coordinate system. The most convenient way is in terms of the direction cosines of the uniform plane wave, the equiphase surfaces are planes perpendicular to the direction of propagation.

Definitions:

uniform planes – a free space plane wave at an infinite distance from the generator, having constant amplitude electric and magnetic field vectors over the equiphase surfaces.

equiphase surface – any surface in a wave over which the field vectors of a particular instant have either 0° or 180° phase difference.

For a plane wave propagating along the +z axis

$$\hat{E}(z) = \hat{E}_m e^{-j\beta z} a_x \qquad (6.1)$$

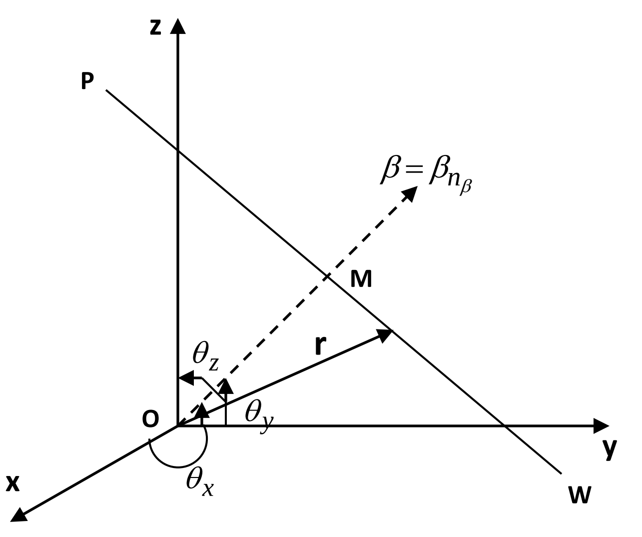
Equation (6.1) states that each z equal to a constant plane will represent an equiphase surface with no spatial variation in the electric or magnetic fields. In other words,

$$\frac{\partial}{\partial x} = 0 = \frac{\partial}{\partial y} \quad \Rightarrow \text{for a uniform plane wave}$$

It will be necessary to replace **z** for a plane wave traveling in an arbitrary direction with an expression when put equal to a constant (βz = constant), that will result in equiphase surfaces.
The equation of an equiphase plane is given by

$$\beta \cdot r = \beta\, n_\beta \cdot r$$

The radial vector (r) from the origin to any point on the plane, and β is the vector normal to the plane is shown in Figure (6.1).



As you can see from figure 6.1, the plane perpendicular to the vector **β** is seen from its side appearing as a line P-W. The dot product $\mathbf{n}_\beta \cdot r$ is the projection of the radial vector **r** along the normal to the plane and will have the constant value **OM** for all points on the plane. The equation **β · r = constant** is the characteristic property of a plane perpendicular to the direction of propagation **β**.

The equiphase equation is

$$\beta \cdot r = \beta_x x + \beta_y y + \beta_z z$$

$$= \beta (\cos \theta_x x + \cos \theta_y y + \cos \theta_z z)$$

$$= \text{constant}$$

$$\mathbf{r} = x a_x + y a_y + z a_z$$

$$\beta = \beta_x a_x + \beta_y a_y + \beta_z a_z$$

θ_x, θ_y, θ_z, are the angles the **β** vector makes with x, y, and z axes, respectively.

Reflection and Refraction at Plane Interface between Two Media:

Figure 6.7 shows two media with electrical properties ε_1 and μ_1 in medium 1, and ε_2 and μ_2 in medium 2. Here a plane wave incident angle θ_i on a boundary between the two media will be partially transmitted into and partially reflected at the dielectric surface. The transmitted wave is reflected into the second medium, so its direction of propagation is different from the incidence wave. The figure also shows two rays for each the incident, reflected, and transmitted waves. A ray is a line drawn normal to the equiphase surfaces, and the line is along the direction of propagation.

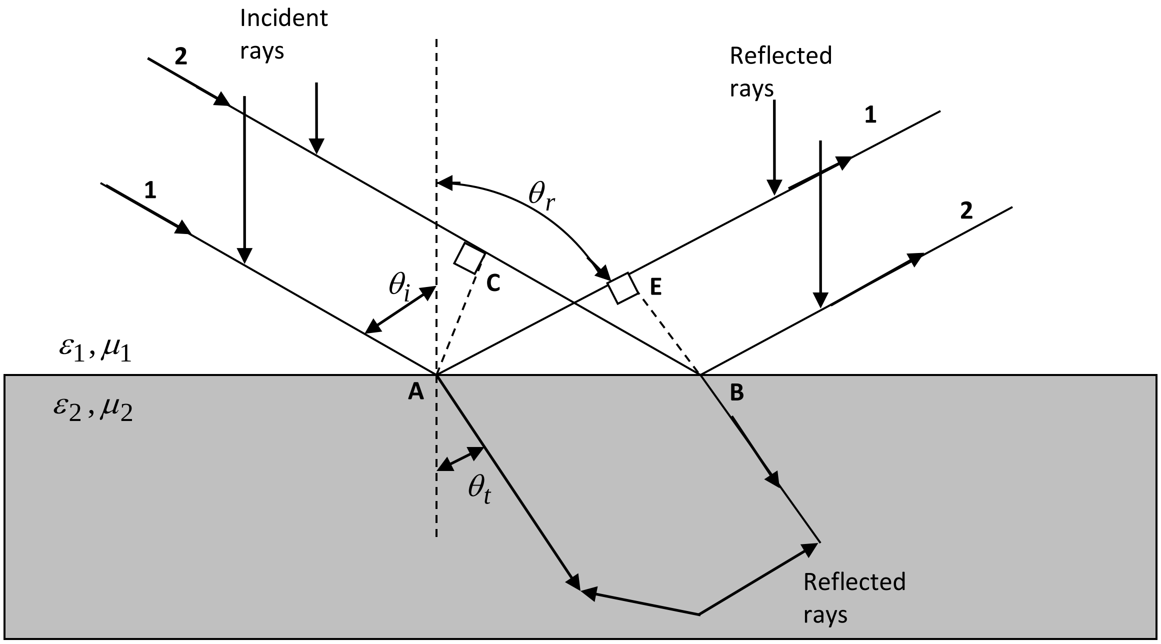


Figure 6.7

The incident ray 2 travels the distance CB, while on the contrary the reflected ray 1 travels the distance AE. For both AC and BE to be the incident and reflected wave fronts or planes of equiphase, the incident wave should take the same time to cover the distance AE. The reason being that the incident and reflected wave rays are located in the same medium, therefore their velocities will be equal,

$$\frac{CB}{V_1} = \frac{AE}{V_2}$$

OR

$$AB \sin\theta_i = AB \sin\theta_r$$

With this being the case then it follows that

$$\theta_i = \theta_r$$

What is the relationship between the angles of incidence θ_i and refraction θr ?

It takes the incident ray the equal amount of time to cover distance CB as it takes the refracted ray to cover distance AD –

$$\frac{CB}{V_1} = \frac{AD}{V_2}$$

And the magnitude of the velocity V_1 in medium 1 is:

$$V_1 = \frac{1}{\sqrt{\mu_1 * \varepsilon_1}}$$

And in medium 2:

$$V_2 = \frac{1}{\sqrt{\mu_2 * \varepsilon_2}}$$

Also,

$$CB = AB \sin\theta_i$$
$$AD = AB \sin\theta_i$$

Therefore,

$$\frac{CB}{AD} = \frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2} = \sqrt{\frac{\mu_2 * \varepsilon_2}{\mu_1 * \varepsilon_1}}$$

For most dielectrics $\mu_2 = \mu_1 = \mu_\circ$

Therefore, $$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\Bigg|_{\mu_1 = \mu_2 = \mu_\circ} \qquad (6.12)$$

Equation 6.12 is known as Snell's Law of Refraction.

Behaviour of Plane waves at the inteface of two media:

We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

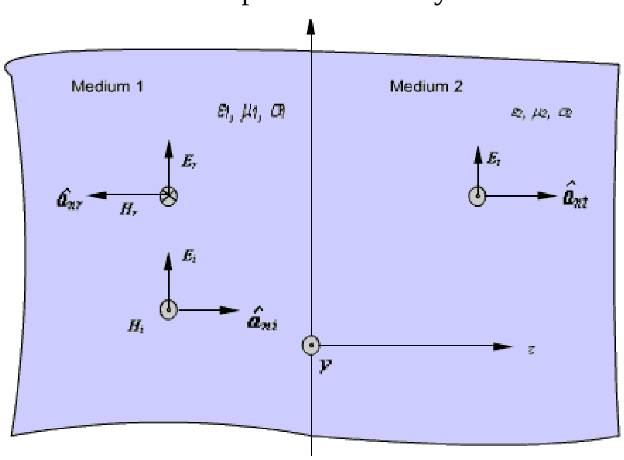


Fig 6 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively. The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium 1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_X \quad \text{.................(1)}$$

$$\vec{H}_i(z) = \frac{1}{\eta_i} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad \text{......................(2)}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\dfrac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at *z* =0, the incident wave will undergo partial reflection and partial transmission.The reflected wave will travel along $\hat{a}_z$ in medium 1.

The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \quad \text{..(3)}$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \quad \text{.........(4)}$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x \quad \text{...(5)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \quad \text{...(6)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1,

$\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2,

$\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$

& $\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$

From equation 3to 6 we get,

$E_{io} + E_{ro} = E_{to}$ ....(7)

$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2}$ ....(8)

Eliminating E_{to} ,

$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}(E_{io} + E_{ro})$

or, $E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$

or, $E_{ro} = \tau E_{io}$

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$ ....(8)

is called the reflection coefficient.

From equation (8), we can write

$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$

or, $E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2}E_{io} = TE_{io}$

$T = \frac{2\eta_2}{\eta_1 + \eta_2}$ ....(9)

is called the transmission coefficient.

We observe that,

$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$ ....(10)

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \le |\tau| \le 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$\therefore \eta_1 = \sqrt{\dfrac{\mu_1}{\in_1}}$

$\eta_2 = 0$

$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\in_1)}$

$= j\omega\sqrt{\mu_1 \in_1} = j\beta_1$

From (9) and (10)

$\tau = -1$

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\hat{a}_x - E_{io}e^{j\beta_1 z}\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$

& $\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$ ....(11)

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$\vec{H}_1(z,t) = \hat{a}_y\dfrac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t$ ....(12)

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given $'t'$, both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

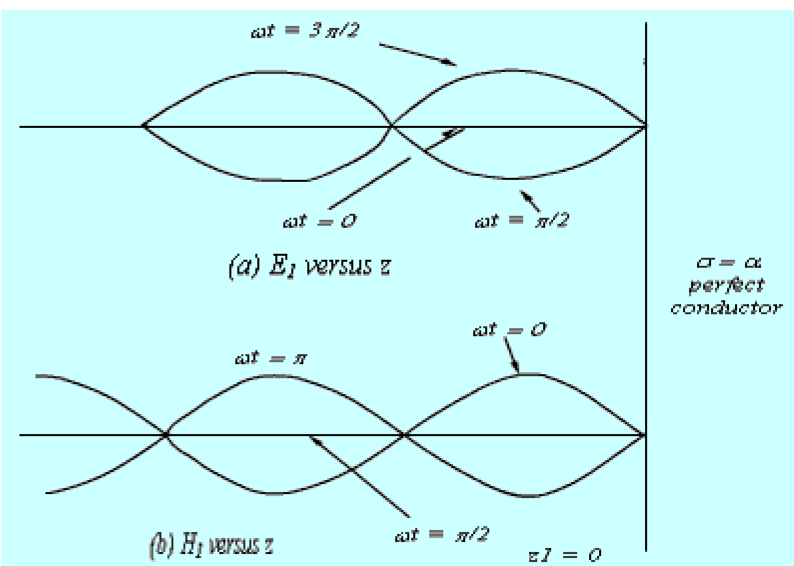


Figure 7: Generation of standing wave

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

$$\left.\right\}\text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

$$\left.\right\}\text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4}, \quad n = 0,1,2,\ldots$$

Case2: Normal incidence on a plane dielectric boundary : If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.

From equation (10) and equation (13) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad \text{..................(14)}$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0, \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad \text{..................(15)}$$

In this case both η_1 and η_2 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)\\ &= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right)\\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma(2j\sin\beta_1 z)\right) \quad \text{..................(16)}\end{aligned}$$

From (6.61), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$. The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows. The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \quad \text{..................(17)}$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{\max} = E_{io}(1+T) \quad \text{..................(18)}$$

and this occurs when

$$2\beta_1 z_{\max} = -2n\pi$$

or $z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$, n = 0, 1, 2, 3......................(19)

The minimum value of $|\vec{E}_1|$ is

$|\vec{E}_1|_{\min} = E_{io}(1-\Gamma)$ ....(20)

And this occurs when

$2\beta_1 z_{\min} = -(2n+1)\pi$

or $z_{\min} = -(2n+1)\frac{\lambda_1}{4}$, n = 0, 1, 2, 3............................(21)

For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $|\vec{E}_1|$ is $E_{io}(1-\Gamma)$ which occurs at the z_{min} locations and the minimum value of $|\vec{E}_1|$ is $E_{io}(1+\Gamma)$ which occurs at z_{max} locations as given by the equations (6.64) and (6.66).

From our discussions so far we observe that $\frac{|E|_{\max}}{|E|_{\min}}$ can be written as

$S = \frac{|E|_{\max}}{|E|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$ ....(22)

The quantity *S* is called as the standing wave ratio.
As $0 \le |\Gamma| \le 1$ the range of *S* is given by $1 \le S \le \infty$

From (6.62), we can write the expression for the magnetic field in medium 1 as

$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$ ....(23)

From (6.68) we find that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.
In medium 2, the transmitted wave propagates in the + z direction.

Brewster Angle:

Brewster angle is defined as the angle of incidence at which there will be no reflected wave. It occurs when the incident wave is polarized such that the **E** field is parallel to the plane of incidence.

Brewster Angle – (from Brewster's Law), the polarizing angle of which (when light is incident) the reflected and refracted index is equal to the tangent of the polarizing angle. In other words, the angle of incidence of which there is no reflection.
From the reflection coefficient expression-

$$\hat{\Gamma}_{||} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

It can be seen that there is an angle of incidence at $\hat{\Gamma}_{||} = 0$. This angle can be obtained when

$$\eta_1 \cos\theta_i = \eta_2 \cos\theta_t$$

Or

$$\cos\theta_i = \frac{\eta_2}{\eta_1}\cos\theta_t$$

The angle of incidence θ_i, at which $\hat{\Gamma}_{||} = 0$, is known as the Brewster angle. The expression for this angle in terms of the dielectric properties of media 1 & 2, considering Snell's Law for the special case $\mu_1 = \mu_2 = \mu_\circ$ is

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\Bigg|_{\mu_1=\mu_2=\mu_\circ}$$

This condition is important, because it is usually satisfied by the materials often used in optical applications.

Equation 6.19 will take the form –

$$\cos\theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\cos\theta_t$$

Square both sides of equation 6.20 and use Snell's Law for the special case of $\mu_1 = \mu_2 = \mu_\circ$ for the following result:

$$\cos^2\theta_i \frac{\varepsilon_1}{\varepsilon_2} = \cos^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_t\right)$$

$$= \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_i\right)$$

The last substitution was based on Snell's Law of refraction. Therefore,

$$\left(1 - \sin^2\theta_i\right) = \frac{\varepsilon_1}{\varepsilon_2} - \frac{\varepsilon_1^2}{\varepsilon_2^2}\sin^2\theta_i$$

$$1-\frac{\varepsilon_1}{\varepsilon_2}=\sin^2\theta_i\left(1-\frac{\varepsilon_1^2}{\varepsilon_2^2}\right)$$

And

$$\sin^2\theta_i=\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}$$

The Brewster angle of incidence is

$$\sin\ \theta_i\ =\sqrt{\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}}$$

A specific value of θ_i can be obtained from equation 6.21 -

$$1-\cos^2\theta_i=\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}$$

Or

$$\cos^2\theta_i=1-\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}=\frac{\varepsilon_1}{\varepsilon_2+\varepsilon_1}=$$

$$\cos\ \theta_i=\sqrt{\frac{\varepsilon_1}{\varepsilon_2+\varepsilon_1}} \qquad (6.23)$$

From equations 6.22 & 6.23 –

$$\tan\theta_i=\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

This specific angle of incidence θ_i is called the Brewster angle θ_β.

$$\theta_\beta=\tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

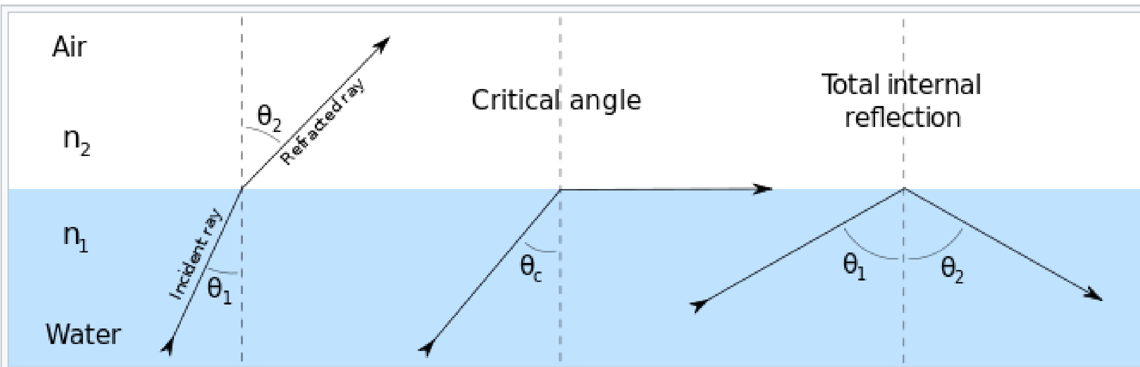
Critical angle:

In geometric optics, at a refractive boundary, the smallest angle of incidence at which total internal reflection occurs. The critical angle is given by

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right) ,$$

Where Θ_c is the critical angle, n_1 is the refractive index of the less dense medium, and n_2 is the refractive index of the denser medium.

Angle of incidence: The angle between an incident ray and the normal to a reflecting or refracting surface



Refraction of light at the interface between two media, including total internal reflection.

Total Reflection at Critical Angle of Incidence

In the previous section it was shown that for common dielectrics, the phenomenon of total transmission exists only where the electric field is parallel to the plane of incidence known as parallel polarization.
There is a second phenomenon existing for both polarizations:

- Total reflection occurring at the interface between two dielectric media
- A wave passing from a medium with a larger dielectric constant to a medium with smaller value of ε

Snell's Law of refraction shows –

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{or} \qquad \sin\theta_i = \frac{\sin\theta_t}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}} \qquad (6.26)$$

Therefore, if $\varepsilon_1 > \varepsilon_2, and\ \theta_t > \theta_i$ then a wave incident at an angle θ_i will pass into medium 2 at a larger angle θ_t .

Definition:

θ_c , (critical angle of incidence) is the value of θ_i that makes $\theta_t = \pi/2$, see Figure 6.13.

Substitute $\theta_t = \pi/2$ in equation 6.26 to get –

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}, \text{ or } \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

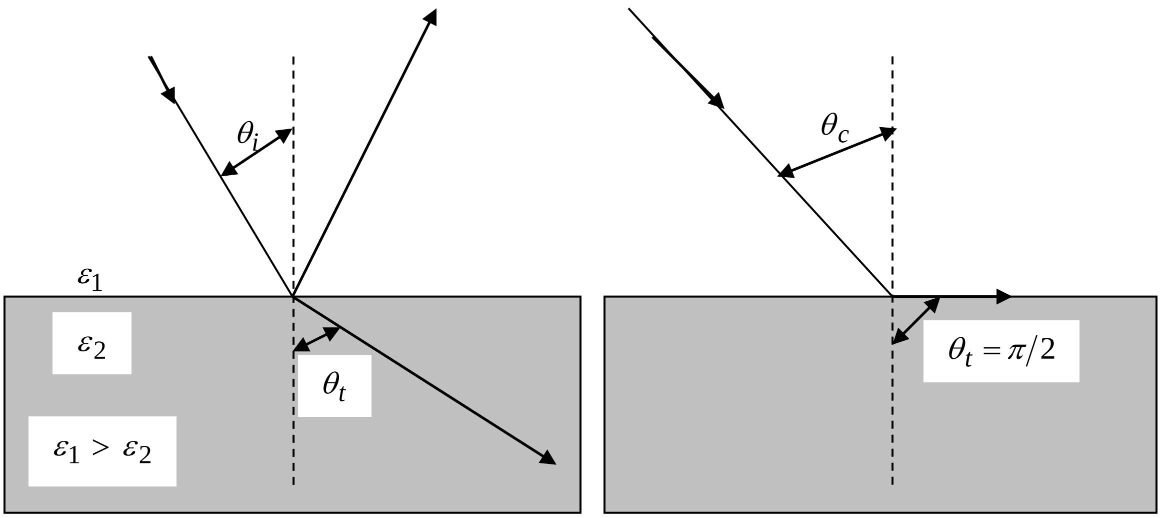


Figure 6.13 illustrates the fact that $\theta_t > \theta_i$, *if* $\varepsilon_1 > \varepsilon_2$. The critical angle θ_c is defined as the value of θ_i at which $\theta_t = \pi/2$.

Envision a beam of light impinging on an interface between two transparent media where $n_i < n_t$. At normal incidence ($\theta_i = 0$) most of the incoming light is transmitted into the less dense medium. As θ_i increases, more and more light is reflected back into the dense medium, while θ_t increases. When $\theta_t = 90°$, θ_i is defined to be θ_c and the transmittance becomes zero. For $\theta_i > \theta_c$ all of the light is totally internally reflected, remaining in the incident medium.

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities asscociated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E} \times \vec{H}\right) = \vec{H}.\nabla \times \vec{E} - \vec{E}.\nabla \times \vec{H}$$

the above curl equations we can write

$$\nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)$$

$$or,\ \nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\vec{J} - \vec{E}.\frac{\partial\vec{D}}{\partial t} \quad\text{..(1)}$$

In simple medium where $\in, \mu$ and σ are constant, we can write

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right) \quad \text{and} \quad \vec{E}.\vec{J} = \sigma E^2$$

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV \quad\text{.........................(2)}$$

The term $\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (6.36) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$ where $\vec{P} = \vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t) = \text{Re}\left[\vec{E}(z)e^{j\omega t}\right] = E_0\cos(\omega t - \beta z)\,\hat{a}_x \quad\text{................................(1)}$$

when E_0 is real.
Let us consider two instanteneous quantities A and B such that

$$A = \mathrm{Re}\left(Ae^{j\omega t}\right) = |A|\cos\left(\omega t + \alpha\right) \qquad ....(2)$$

$$B = \mathrm{Re}\left(Be^{j\omega t}\right) = |B|\cos\left(\omega t + \beta\right)$$

where A and B are the phasor quantities.

i.e, $A = |A|e^{j\alpha}$

$B = |B|e^{j\beta}$

Therefore,

$$AB = |A|\cos\left(\omega t + \alpha\right)|B|\cos\left(\omega t + \beta\right)$$

$$= \frac{1}{2}|A||B|\left[\cos\left(\alpha - \beta\right) + \cos\left(2\omega t + \alpha + \beta\right)\right] \qquad ....(3)$$

Since A and B are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos\left(\alpha - \beta\right) \qquad ....(4)$$

Further, considering the phasor quantities A and B, we find that

$$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}\left(AB^*\right) = |A||B|\cos\left(\alpha - \beta\right)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}\left(AB^*\right) \qquad ....(5)$$

The poynting vector $\vec{P} = \vec{E}\times\vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x\left(E_yH_z - E_zH_y\right) + \hat{a}_y\left(E_zH_x - E_xH_z\right) + \hat{a}_z\left(E_xH_y - E_yH_x\right) \qquad ....(6)$$

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (6.42) we can write:

$$\vec{P}_z = E_x\left(z,t\right)H_y\left(z,t\right)\hat{a}_3$$

Using (6)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x\left(z\right)H_y^*\left(z\right)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z) \times H_y(z)\right) \qquad \text{.......................................(7)}$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.

For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right) \qquad \text{.....................(8)}$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}\left(\vec{S}\right)$.

Polarization of plane wave:

The polarization of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficient as the magnetic field components are related to electric field vector by the Maxwell's equations.

Let us consider a plane wave travelling in the +z direction. The wave has both E_x and E_y components.

$$\vec{E} = \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$$

The corresponding magnetic fields are given by,

$$\vec{H} = \frac{1}{\eta}\hat{a}_z \times \vec{E}$$

$$= \frac{1}{\eta}\hat{a}_z \times \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$$

$$= \frac{1}{\eta}\left(-E_{oy}\hat{a}_x + E_{ox}\hat{a}_x\right) e^{-j\beta z}$$

Depending upon the values of E_{ox} and E_{oy} we can have several possibilities:

1. If $E_{oy} = 0$, then the wave is linearly polarised in the x-direction.

2. If $E_{oy} = 0$, then the wave is linearly polarised in the y-direction.

3. If E_{ox} and E_{oy} are both real (or complex with equal phase), once again we get a linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1}\frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig1 below

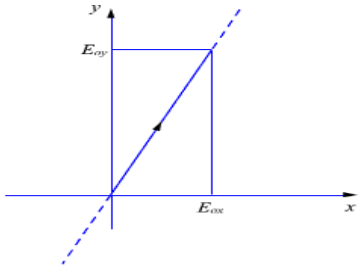


Fig1 : Linear Polarisation

4. If *Eox* and *Eoy* are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

Let $E_{ox} = |E_{ox}| e^{ja}$

$E_{oy} = |E_{oy}| e^{jb}$

Then,

$E_x(z,t) = \mathrm{Re}\left[|E_{ox}| e^{ja} e^{-j\beta z} e^{j\omega t}\right] = |E_{ox}| \cos(\omega t - \beta z + a)$

and $E_y(z,t) = \mathrm{Re}\left[|E_{oy}| e^{jb} e^{-j\beta z} e^{j\omega t}\right] = |E_{oy}| \cos(\omega t - \beta z + b)$ ....(2)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (2) we find that,

$E_x(o,t) = |E_{ox}| \cos \omega t$

$E_y(o,t) = |E_{oy}| \cos\left(\omega t + \frac{\pi}{2}\right) = |E_{oy}|(-\sin \omega t)$

$\therefore \left(\frac{E_x(o,t)}{|E_{ox}|}\right)^2 + \left(\frac{E_y(o,t)}{|E_{oy}|}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1$ ....(3)

and the electric field vector at z = 0 can be written as

$\vec{E}(o,t) = |E_{ox}| \cos(\omega t) \hat{a}_x - |E_{oy}| \sin(\omega t) \hat{a}_y$ ....(4)

Assuming $|E_{ox}| > |E_{oy}|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 2

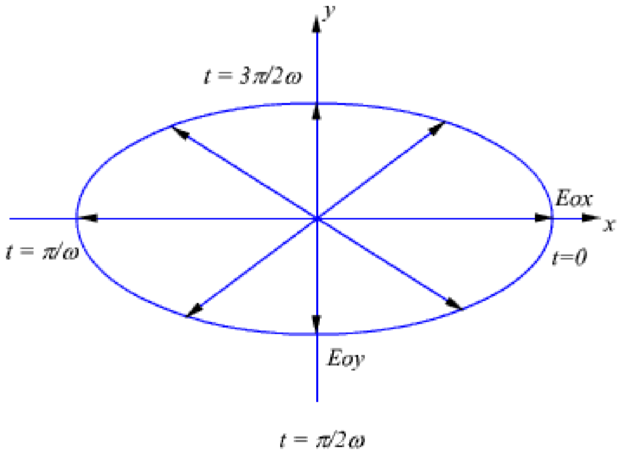


Figure 2 : Plot of *E*(*o*,*t*)

From equation (6.47) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces qn ellipse and the field is said to be elliptically polarised.

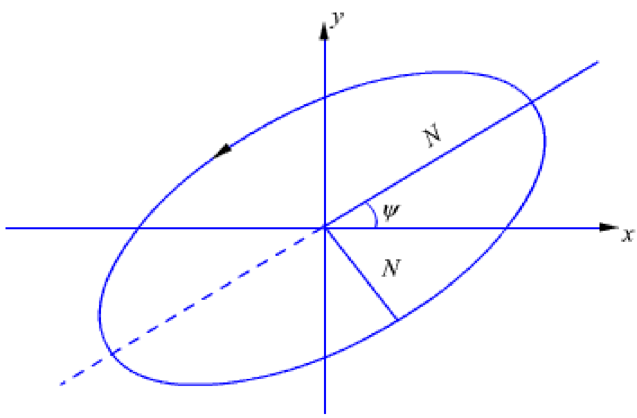


Figure 3: Polarisation ellipse

The polarisation ellipse shown in figure 3 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle ψ(orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW). Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity

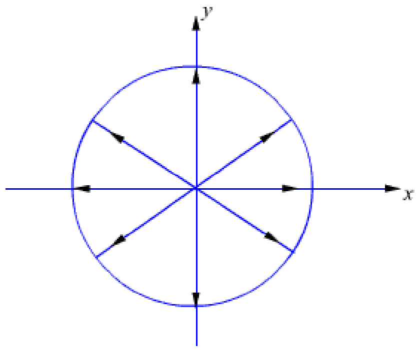


Figure 5: Circular Polarisation (RHCP)

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground(vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation (one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

Solved Problems:

1. Calculate the polarization angle (Brewster angle) for an air water $\left(\varepsilon_r = 81\right)$ interface at which plane waves pass from the following:
   (a) Air into water.
   (b) Water into air.

SOLUTION

1. (a) Air into water:

$$\varepsilon_{r1} = 1 \quad and \quad \varepsilon_{r2} = 81$$

The Brewster angle is then given by

$$\theta_\beta = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 6.34°$$

Therefore,

$$\theta_\beta = \tan^{-1}\sqrt{81} = 83.7°$$

(b) Water into air:

$$\varepsilon_{r1} = 81 \quad and \quad \varepsilon_{r2} = 1$$

Hence,

$$\theta_\beta = \tan^{-1}\sqrt{\frac{1}{81}} = 6.34°$$

To relate the Brewster angles in both cases, let us calculate the angle of refraction.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Therefore, in case a,

$$\frac{\sin\theta_B}{\sin\theta_t} = \sqrt{81}$$

Therefore,

$$\sin\theta_t = \frac{\sin 83.7}{9} = 0.11$$

Or $\theta_t = 6.34^\circ$, which is the same as the Brewster angle for case b. Also, the angle of refraction in case b is given by Snell's Law as:

$$\frac{\sin\theta_B}{\sin\theta_t} = \sqrt{\frac{\varepsilon_\circ}{81\varepsilon_\circ}} = \sqrt{\frac{1}{81}}$$

Therefore,

$$\sin\theta_t = \frac{\sin 6.34°}{\sqrt{\frac{1}{81}}} = 0.99$$

Or $\theta_t = 83.7°$, which is the Brewster angle for case a.

2. The index of refraction of liquid is 1.9. What is the critical angle for a light ray travelling in the liquid toward a flat layer of air?

Solution

The critical angle is determined by the following expression (Snell's law, in which the angle of refraction is 90^0):

$$n_1 \sin\theta_{cr} = n_2 \sin 90^0$$

Here $n_1 = 1.9$ is the index of refraction of medium 1 (liquid), $n_2 = 1$ is the index of refraction of medium 2 (air). We substitute the known values in the above expression and find the critical angle

$$\sin\theta_{cr} = \frac{n_2}{n_1} = \frac{1}{1.9}$$

$$\theta_{cr} = \sin^{-1}\frac{1}{1.9} = 37.36^0$$

3. Find the critical angle for total internal reflection for light going from ice (index of refraction = 1.31) into air.

Solution

The critical angle is defined as the angle of incidence for which the corresponding angle of refraction is 90^0. Then the Snell's law takes the following form

$$n_1 \sin\theta_{cr} = n_2 \sin\theta_2$$

Here $n_1 = 1.31$ is the index of refraction of medium 1 (ice), θ_{cr} is the unknown critical angle, $\theta_2 = 90^0$ is the angle of refraction (angle in air), and $n_2 = 1$ is the index of refraction of medium 2 (air). We substitute these values into above expression and obtain

$$1.31 \sin\theta_{cr} = 1 \sin 90^0 = 1$$

Then

$$\theta_{cr} = \sin^{-1}\frac{1}{1.31} = 49.76^0$$