# UNIT - V

# MULTIRATE DIGITAL SIGNAL PROCESSING

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Multirate digital signal processing (MDSP) is required in digital systems when more than one sampling rate is required.

"The discrete time systems that process data at more than one sampling rate are known as "Multirate systems".

For example, In digital audio, different sampling rates ~~are required~~ used ~~for~~ are 32KHz for broadcasting, 44.1 KHz for compact disc and 48KHz for audio tape.

In digital video, the sampling rates used are 14.318 MHz for NTSC and 17.73 MHz for PAL.

The basic operations in MDSP are decimation and Interpolation.

* Decimation is used for reducing the sampling rate.
* Interpolation is used for increasing the sampling rate.

Advantages of MDSP:-

1. Computational requirements are less.
2. storage for filter coefficients are less.
3. Finite arithmetic effects are less.
4. Filter order required in multirate applications are low.
5. Sensitivity to filter coefficient lengths are less.

* NTSC - National Television System Committee
* PAL - phase Alternating line




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## Sampling :-

Let $x(t)$ is a continuous - time varying signal. The signal $x(t)$ is sampled at regular interval of time with sampling period 'T'.

The sampled signal $x(nT)$ is given as

$$x(nT) = x(t)\Big|_{t=nT} \longrightarrow (1)$$

The sampling process can also be interpreted as a modulation or multiplication process as depicted in fig.1 The continuous time signal $x(t)$ is multiplied by the sampling function $s(t)$ which is a series of impulses. The resultant signal will be a discrete - time signal $x(n)$ i.e.,

$$x(n) = x(t) \cdot s(t)\Big|_{t=nT,\ -\infty < n < \infty} \longrightarrow (2)$$

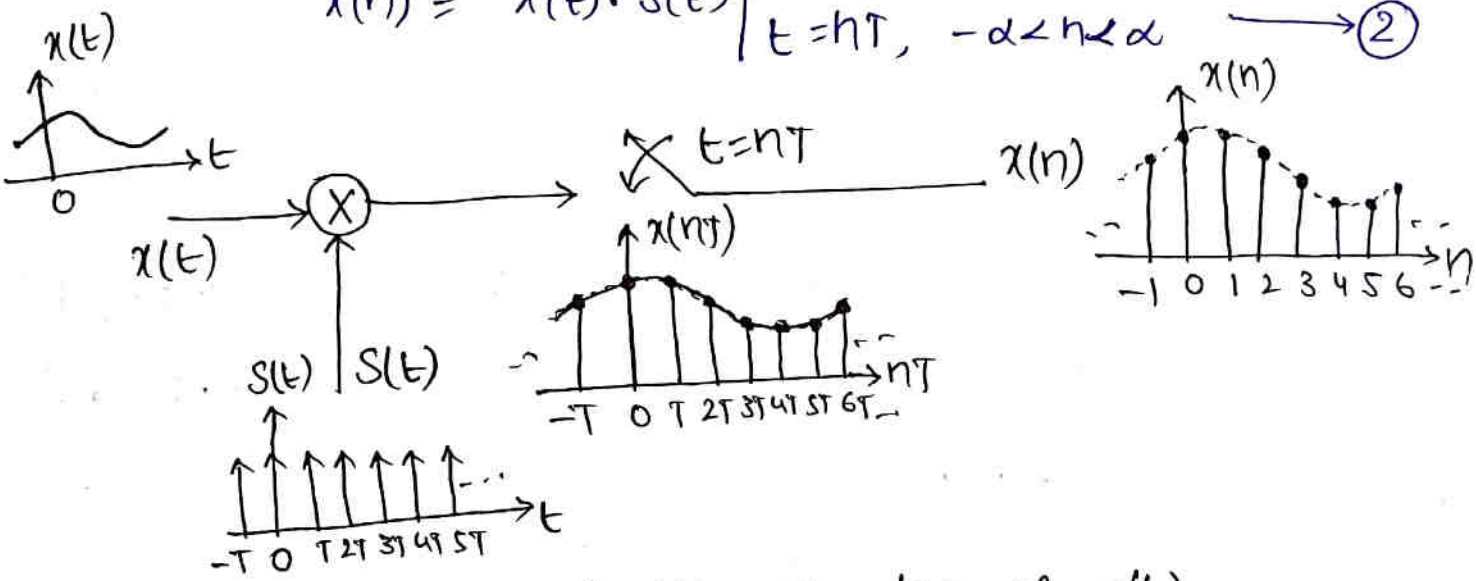



Fig. 1 :- Periodic sampling of $x(t)$

## Sampling theorem :-

A band limited signal $x(t)$ having a finite energy, which has no frequency components higher than $f_h$ hertz can completely described and reconstructed from its samples taken at the rate of $2f_h$ samples per second.


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## Decimation (or) Down sampling :-

The process of reducing the sampling rate by an integer factor (D) is called "Decimation" of the sampling rate. It is also called "Down sampling" by factor (D).

Fig.2 shows the block diagram of a decimator.

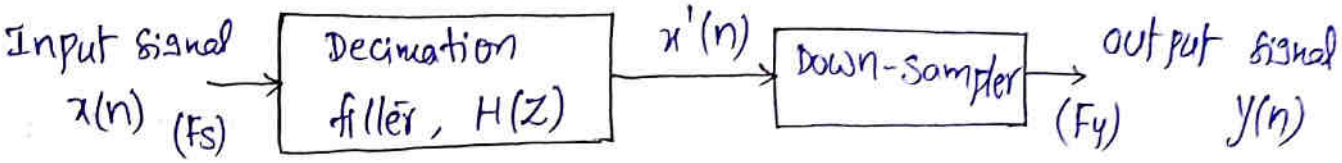



Fig. 2 :- Block diagram of a decimator.

This decimator comprises of two blocks such as decimation filter and down sampler.

- Decimation filter is used to band limit the signal before decimation operation.
- Down sampler decreases the sampling rate by an integer factor (D).

### Down sampler :-

The op signal $y(n)$ is a down sampled signal of the input signal $x(n)$ and can be represented by

$$y(n) = x(Mn) \longrightarrow (3)$$

Where 'M' is down sampled factor.

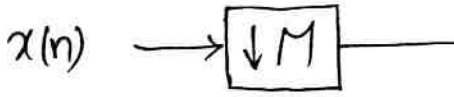
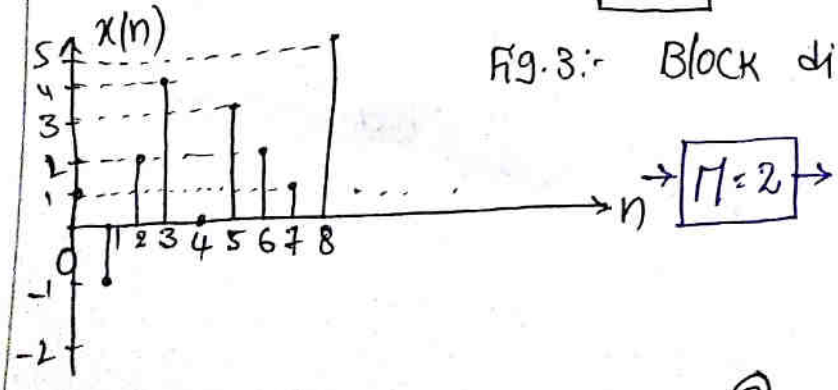



Fig.3 :- Block diagram of a down sampler







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## Decimation of sampling rate by an integer factor (D)

(or)

## Down sampling by a factor 'D' :-

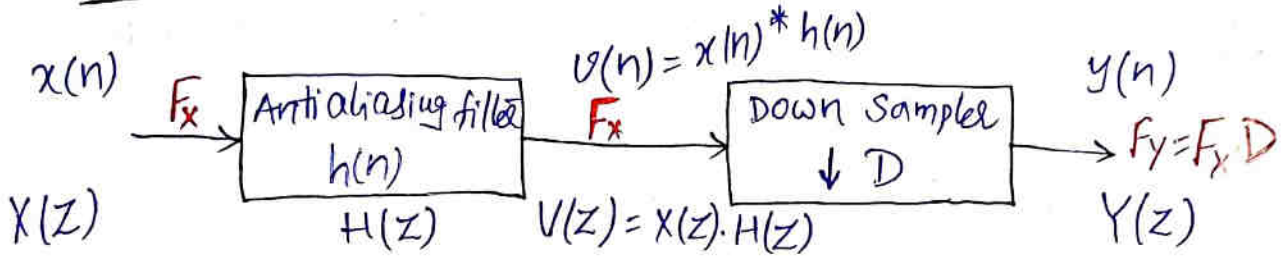



Fig: Decimation by a factor 'D'

→ Let us consider that $x(n)$ is having a spectrum $X(\omega)$ with $0 \le \omega \le \pi$. Let the sampling frequency '$f_x$' and maximum frequency '$f_{max}$' be related as

$$f_{max} \le \frac{f_x}{2} \quad \Rightarrow \quad f_x \ge 2 f_{max}$$

→ If we reduce the sampling rate by 'D' i.e., $\frac{f_x}{D}$ then maximum frequency will be $f_{max} = \frac{f_x}{2D}$. Thus maximum frequency is changed from $\frac{f_x}{2}$ to $\frac{f_x}{2D}$. this is called "Aliasing".

→ To avoid aliasing, $x(n)$ is band limited to $|\omega| \le \frac{\pi}{D}$. Then this band limited signal is downsampled by the factor 'D'.

From fig, $v(n) = x(n) * h(n) = \sum_{k=0}^{\infty} h(k).x(n-k)$ ——→ (1)

The o/p $y(n)$ can be written as

$y(n) = v(nD)$ [∵ $v(n)$ is downsampled by factor 'D']

$\Rightarrow y(n) = \sum_{k=0}^{\infty} h(k).x(nD-k)$ ——→ (2)


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The down sampled sequence $y(n)$ is picks up only at every $D^{th}$ sample. So, the down sampled sequence is given by product of $v(n)$ with sampling sequence $P(n)$.

$$\tilde{v}(n) = v(n) \cdot P(n) \longrightarrow (3)$$

Where $P(n)$ can be represented by discrete fourier series

i.e., $$P(n) = \frac{1}{D} \sum_{k=0}^{D-1} e^{j\frac{2\pi kn}{D}} \longrightarrow (4)$$

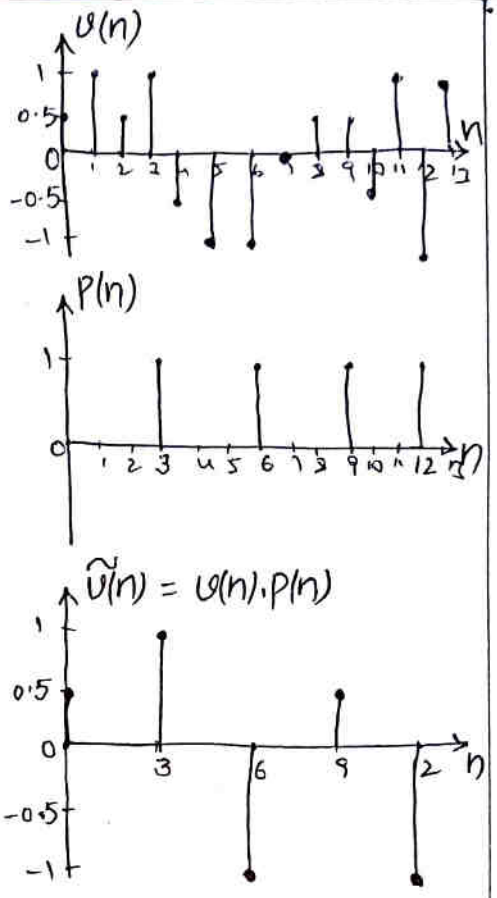



Fig:- Down sampling process

Eqn (2) will be updated as

$$y(n) = \tilde{v}(nD) \longrightarrow (5)$$

Applying Z-transform to this equation

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} \tilde{v}(nD) z^{-n} \quad [\because eqn (5)]$$

$$\Rightarrow Y(z) = \sum_{n=-\infty}^{\infty} \tilde{v}(n) z^{-n/D}$$

$$= \sum_{n=-\infty}^{\infty} v(n) \cdot P(n) z^{-n/D} \quad [\because eqn (3)]$$

$$= \sum_{n=-\infty}^{\infty} v(n) \left[ \frac{1}{D} \cdot \sum_{k=0}^{D-1} e^{j\frac{2\pi kn}{D}} \right] \cdot z^{-n/D}$$

Interchanging the order of summation, $[\because eqn (4)]$




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$$Y(z) = \frac{1}{D}\sum_{K=0}^{D-1}\sum_{n=-\alpha}^{\alpha} v(n).\, e^{\frac{j2\pi Kn}{D}}.z^{-n/D}$$

$$= \frac{1}{D}\sum_{K=0}^{D-1}\left[\sum_{n=-\alpha}^{\alpha} v(n)\left[e^{\frac{-j2\pi K}{D}}.z^{\frac{1}{D}}\right]^{-n}\right]$$

The quantity [·] indicates Z-transform definition.

$$Y(z) = \frac{1}{D}\sum_{K=0}^{D-1} V\left[e^{\frac{-j2\pi K}{D}}.z^{1/D}\right] \longrightarrow (6)$$

$$\Rightarrow Y(z) = \frac{1}{D}\sum_{K=0}^{D-1} X\left[e^{-j\frac{2\pi K}{D}}.z^{1/D}\right].H\left[e^{-j\frac{2\pi K}{D}}.z^{1/D}\right]$$

$[\because V(z) = X(z).H(z)]$

If $Y(z)$ is evaluated on unit circle then we get spectrum of $y(n)$ by putting $z = e^{jw_y}$. Here $w_y$ is frequency of $y(n)$.

$$Y(e^{jw_y}) = \frac{1}{D}\sum_{K=0}^{D-1} X\left[e^{\frac{-j2\pi K}{D}}.e^{\frac{jw_y}{D}}\right].H\left[e^{\frac{-j2\pi K}{D}}.e^{\frac{jw_y}{D}}\right]$$

$$\Rightarrow Y(e^{jw_y}) = \frac{1}{D}\sum_{K=0}^{D-1} X\left[e^{j\left(\frac{w_y-2\pi K}{D}\right)}\right].H\left[e^{j\left(\frac{w_y-2\pi K}{D}\right)}\right] \longrightarrow (7)$$

Normally, we write

$$Y(w_y) = \frac{1}{D}\sum_{K=0}^{D-1} X\left(\frac{w_y-2\pi K}{D}\right).H\left(\frac{w_y-2\pi K}{D}\right) \longrightarrow (8)$$

From above eqn. $X\left(\frac{w_y-2\pi K}{D}\right)$ indicates replica of $X\left(\frac{w_y}{D}\right)$ at $0, 2\pi, 4\pi, \ldots 2\pi(D-1)$.


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At $K=0$,

$$Y(\omega_y) = \frac{1}{D} \cdot X\left(\frac{\omega_y}{D}\right) \cdot H\left(\frac{\omega_y}{D}\right) \longrightarrow (9)$$

W.K.T, anti aliasing filter band limits the signal to $\frac{\pi}{D}$,

i.e.. $$H(\omega) = \begin{cases} 1 & : |\omega| \le \frac{\pi}{D} \\ 0 & : \text{elsewhere} \end{cases}$$

$$H\left(\frac{\omega_y}{D}\right) = \begin{cases} 1 & : \left|\frac{\omega_y}{D}\right| \le \frac{\pi}{D} \quad (\text{or}) \ |\omega_y| \le \pi \\ 0 & : \text{elsewhere} \end{cases} \longrightarrow (10)$$

Eqn. (9) becomes

$$\boxed{Y(\omega_y) = \frac{1}{D} \cdot X\left(\frac{\omega_y}{D}\right) = \frac{1}{D} \cdot X(\omega_x)} \quad \left[\because \omega_x = \frac{\omega_y}{D}\right] \longrightarrow (11)$$

for $0 \le |\omega_y| \le \pi$

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Eg:- A signal $x(n)$ at a sampling frequency $f_o$ 2.048 KHz is to be decimated by a factor of 32 to yield a signal at sampling frequency of 64Hz. The signal band of interest extends from 0 to 30Hz. The anti-aliasing filter should satisfy the following specifications:

Pass band deviation : 0.01dB  Passband : 0-30Hz
Stop band deviation : 80dB  Stopband : 32-64Hz

The signal components in the range from 30 to 32Hz should be protected from aliasing. Design a suitable one-stage decimator.



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Sol:-

Given that,

Pass band deviation, $\delta p = 0.01dB = 0.001152$ $\left[\because \delta p\,dB = 20.\log(1+\delta p)\right]$

stop band deviation, $\delta s = 80dB = 0.0001$ $\left[\because \delta s\,dB = -20.\log \delta s\right]$

Pass band : $0 \le F \le 30Hz$

stop band : $32 \le F \le 64Hz$

Sampling rate $(f_0) = 2048Hz$ : $D = 32$

output Sampling rate $(fy) = 64Hz$

→ Calculate transition band, $\Delta f = \dfrac{fs - fp}{f_0} = \dfrac{32-30}{2048}$

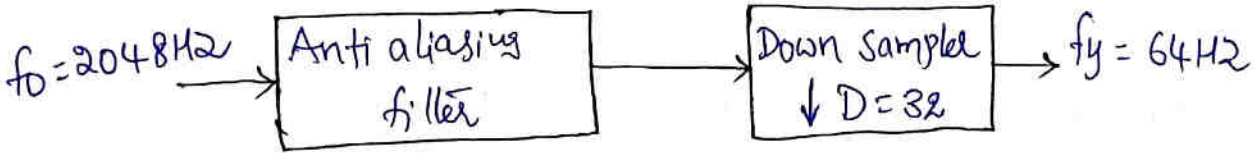
$\Rightarrow \Delta f = 0.0009765$

→ Obtain the order of filter,

$$N = \frac{-10.\log_{10}(\delta p \delta s) - 15}{14.\Delta f} + 1$$

$$\Rightarrow N = \frac{-10.\log_{10}(0.001152 \times 0.0001) - 15}{14 \times 0.0009765} + 1 = 3979.$$

Fig. shows the single stage decimator.



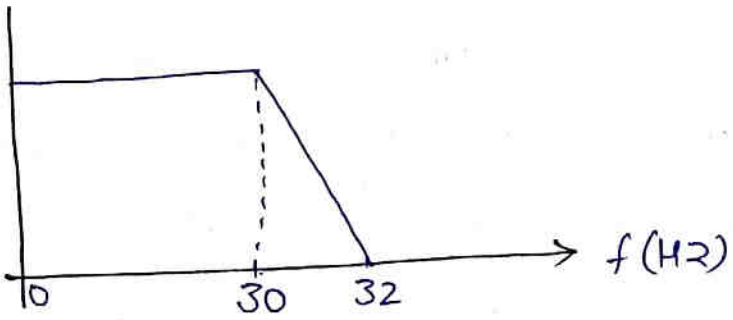




Fig:- Single-stage decimator.

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## Interpolation (or) upsampling :-

The process of increasing sampling rate of a signal is "interpolation" (or) "sampling rate expansion" (or) "upsampling".

Let 'I' be the interpolation factor then the interpolator simply puts (I-1) zeros between successive samples of $x(n)$.

Fig. shows the block diagram of interpolation.

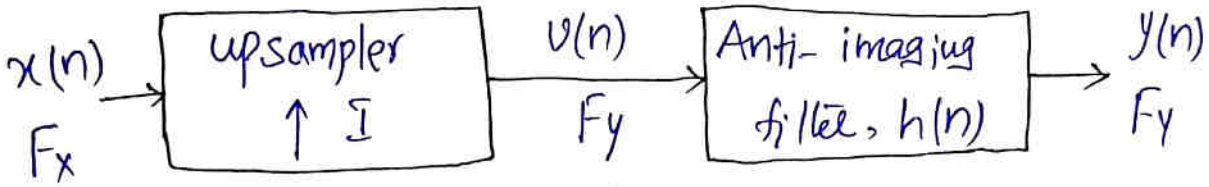



Fig. Interpolation by factor 'I'

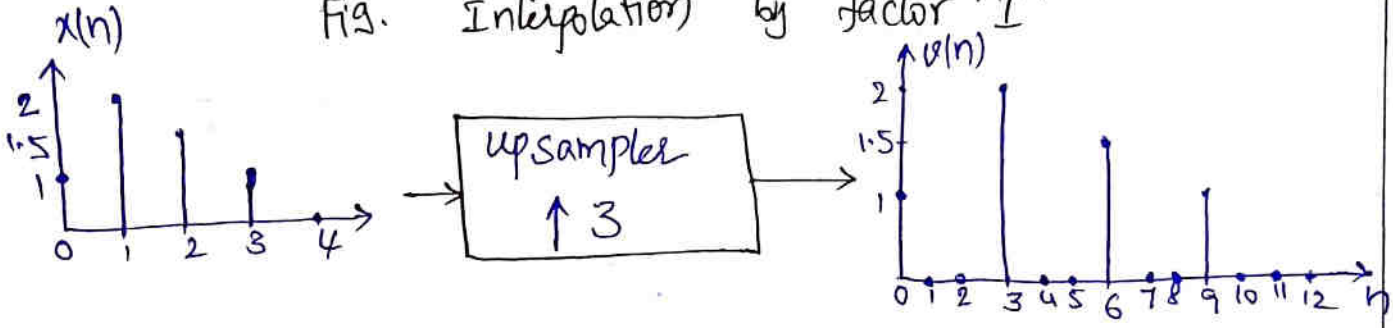



Fig: upsampler

## Interpolation by factor 'I' :-

The output $y(n)$ of anti-imaging filter is given as convolution of $v(n)$ & $h(n)$ i.e.,

$$y(n) = \sum_{K=-\infty}^{\infty} h(n-K).v(K) \longrightarrow (1)$$

Here $h(n-K).v(K) = \begin{cases} \text{Non zero} & : \text{If 'K' is integer multiple of 'I'} \\ 0 & : \text{If 'K' is not integer multiple of 'I'.} \end{cases} \longrightarrow (2)$




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Hence, we can replace 'K' by 'KI' in above equation.

$$y(n) = \sum_{K=-\infty}^{\infty} h(n-KI).\, v(KI) \longrightarrow (3)$$

We know that, $v(KI) = x(K) \longrightarrow (4)$

from (3) & (4),

$$y(n) = \sum_{K=-\infty}^{\infty} h(n-KI).\, x(K) \longrightarrow (5)$$

This is an equation of output signal.

Eqn (4) can also written as

$$v(n) = x\left(\frac{n}{I}\right) \quad ; \quad n = 0, \pm I, \pm 2I, \ldots \longrightarrow (6)$$

Taking Z-transform,

$$V(Z) = \sum_{n=-\infty}^{\infty} v(n) Z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{I}\right) Z^{-n}$$

$$\Rightarrow V(Z) = \sum_{n=-\infty}^{\infty} x(n).\, Z^{-nI} = \sum_{n=-\infty}^{\infty} x(n).\, \left(Z^{I}\right)^{-n}$$

$$\therefore \Rightarrow V(Z) = X(Z^I) \longrightarrow (7)$$

If $V(Z)$ is evaluated on unit circle by putting $Z = e^{jw_y}$

$$V\left(e^{jw_y}\right) = X\left(e^{jw_y I}\right)$$

simply, $V(w_y) = X(w_y I) \longrightarrow (8)$

From block diagram, we know that,

$$w_y = \frac{w_x}{I}$$


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→ Fig. (a) shows an arbitrary spectrum of $x(n)$. Note that the spectrum of $x(n)$ has frequencies from $-\pi$ to $+\pi$ i.e., $-\pi \le \omega_x \le \pi$

→ Fig. (b) shows the spectrum of $v(n)$.

→ Fig. (c) shows the anti-imaging filter response.

$$H(\omega_y) = \begin{cases} C & : -\frac{\pi}{I} \le \omega_y \le \frac{\pi}{I} \\ 0 & : \text{otherwise} \end{cases} \longrightarrow (9)$$

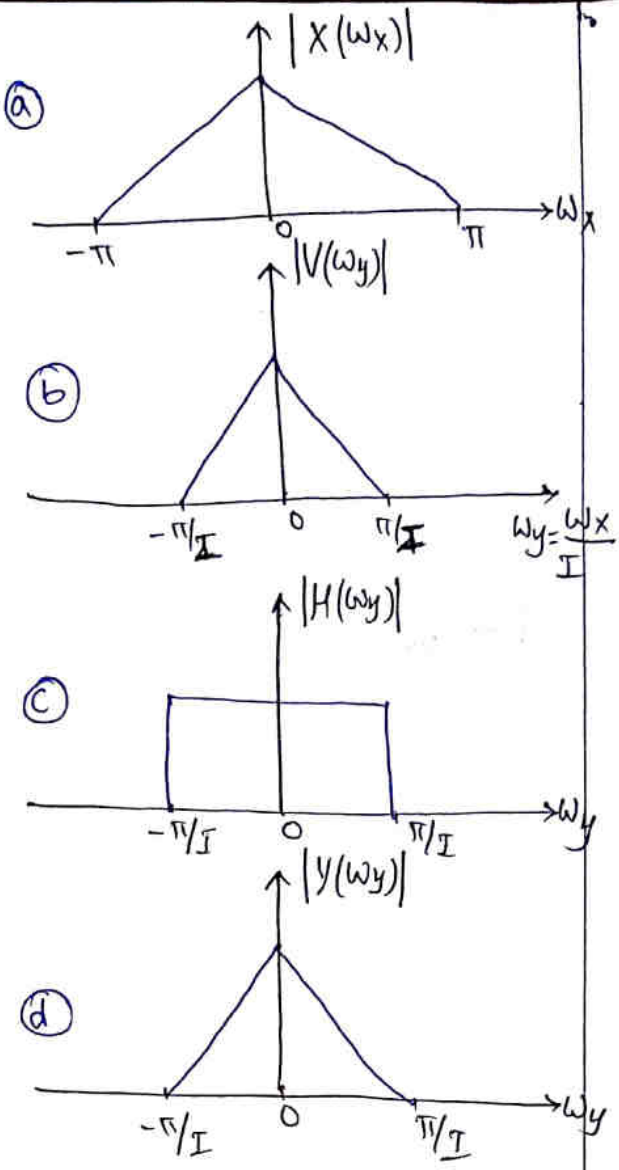



Fig :- Spectrum of signals in upsampling process

Now,

$$Y(\omega_y) = H(\omega_y) \cdot V(\omega_y)$$

$$= H(\omega_y) \cdot X(\omega_y I) \quad (\because \text{from } (8))$$

$$Y(\omega_y) = \begin{cases} C \cdot X(\omega_y \cdot I) & : -\frac{\pi}{I} \le \omega_y \le \frac{\pi}{I} \\ 0 & : \text{otherwise} \end{cases} \longrightarrow (10)$$

Calculate "C" :-

W.K.T, $$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega_y) e^{j\omega_y n} d\omega_y$$

Let $n = 0$, $$y(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega_y)\, d\omega_y$$

$$= \frac{1}{2\pi} \int_{-\pi/I}^{\pi/I} C \cdot X(\omega_y I) \cdot d\omega_y \quad (\because \text{from } (10))$$

(6)


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$$\Rightarrow y(0) = \frac{C}{2\pi} \int_{-\pi}^{\pi} X(\omega_x) \cdot \frac{1}{I} \cdot d\omega_x$$

$$= \frac{C}{I} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega_x) \cdot d\omega_x \right]$$

$$\Rightarrow y(0) = \frac{C}{I} \cdot x(0)$$

Thus, $y(0) = x(0)$, if $C = I$ ⟶ (11)

from (10) & (11),

$$Y(\omega_y) = \begin{cases} I \cdot X(\omega_y \cdot I) & : \frac{-\pi}{I} \le \omega_y \le \frac{\pi}{I} \\ 0 & : \text{otherwise} \end{cases}$$

⟶ (12)

(or)

$$Y(\omega_y) = \begin{cases} I \cdot X(\omega_x) & : -\pi \le \omega_x \le \pi \\ 0 & : \text{otherwise} \end{cases}$$

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Eg: Consider the signal $x(n) = a^n \cdot u(n)$, $|a| < 1$

i). Determine the spectrum of the signal.

ii). The signal is applied to an interpolator that increases sampling rate by a factor of 2. Determine its o/p spectrum.

iii) Show that the spectrum in part (ii) is simply fourier transform of $x(n/2)$.

Sol:- Given that, $x(n) = a^n \cdot u(n)$ , $|a|<1$

i). taking Z-transform,

$$X(Z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$\Rightarrow X(Z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{Z}{Z-a}, \quad |Z|>a$$

Substitute $Z = e^{j\omega}$,

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega}-a} = \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - a}$$

$$\Rightarrow X(e^{j\omega}) = \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega}$$

$$\therefore |X(e^{j\omega})| = \frac{1}{\sqrt{(\cos\omega - a)^2 + \sin^2\omega}} = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

| $\omega$ | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ | $5\pi/4$ | $3\pi/2$ | $7\pi/4$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert X(e^{j\omega})\rvert$ | $\frac{1}{1-a}$ | $\frac{1}{\sqrt{1-\sqrt{2}a+a^2}}$ | $\frac{1}{\sqrt{1+a^2}}$ | $\frac{1}{\sqrt{1+\sqrt{2}a+a^2}}$ | $\frac{1}{1+a}$ | $\frac{1}{\sqrt{1+\sqrt{2}a+a^2}}$ | $\frac{1}{\sqrt{1+a^2}}$ | $\frac{1}{\sqrt{1-\sqrt{2}a+a^2}}$ | $\frac{1}{1-a}$ |

The frequency spectrum of the signal $x(n)$ as shown in fig.

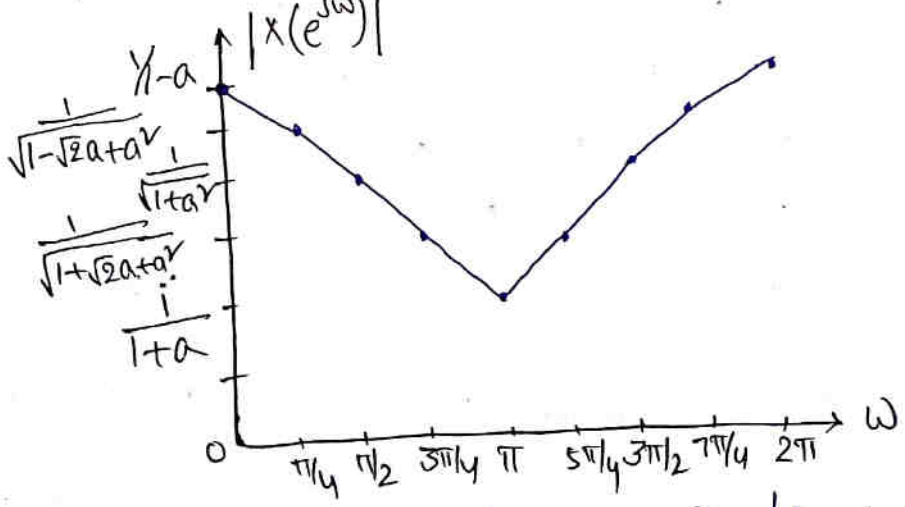



Fig:- Magnitude spectrum of $X(\omega)$

(7)


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ii) The interpolated signal $y(n)$ is obtained by increasing the sampling rate by a factor of '2' for $a^n \cdot u(n)$

Therefore, $y(n) = x(n/2) = a^{n/2} \cdot u(n)$

$$\therefore Y(z) = \sum_{n=0}^{\infty} a^{n/2} z^{-n} = \sum_{n=0}^{\infty} \left(a^{+1/2} z^{-1}\right)^n$$

$$\Rightarrow Y(z) = \frac{1}{1-\sqrt{a}z^{-1}} = \frac{z}{z-\sqrt{a}}, \quad |z| > \sqrt{a}$$

Substitute $z = e^{j\omega}$

$$Y(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega}-\sqrt{a}} = \frac{\cos\omega + j\sin\omega}{(\cos\omega-\sqrt{a}) + j\sin\omega}$$

$$\therefore |Y(e^{j\omega})| = \frac{1}{\sqrt{(\cos\omega-\sqrt{a})^2+\sin^2\omega}} = \frac{1}{\sqrt{1+a-2\sqrt{a}\cdot\cos\omega}}$$

| $\omega$ | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ | $5\pi/4$ | $3\pi/2$ | $7\pi/4$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert Y(e^{j\omega})\rvert$ | $\frac{1}{1-\sqrt{a}}$ | $\frac{1}{\sqrt{1-\sqrt{2}a+a}}$ | $\frac{1}{\sqrt{1+a}}$ | $\frac{1}{\sqrt{1+\sqrt{2}a+a}}$ | $\frac{1}{1+\sqrt{a}}$ | $\frac{1}{\sqrt{1+\sqrt{2}a+a}}$ | $\frac{1}{\sqrt{1+a}}$ | $\frac{1}{\sqrt{1-\sqrt{2}a+a}}$ | $\frac{1}{1-\sqrt{a}}$ |

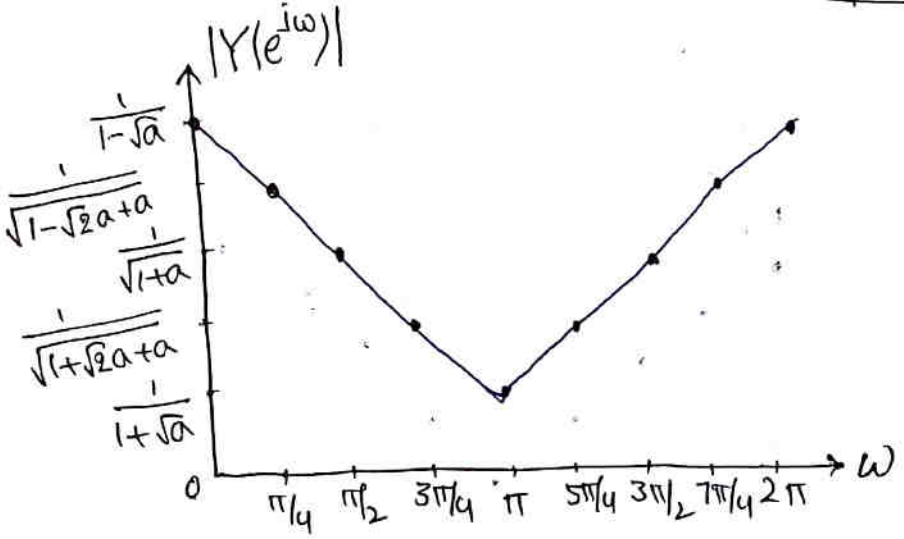



Fig:- Magnitude Spectrum of $y(n) = x(n/2)$


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iii) The fourier transform for $x(n/2)$ is obtained as:

$$F[x(n/2)] = \sum_{n=0}^{\infty} a^{n/2} e^{-j\omega n} = \sum_{n=0}^{\infty} (a^{1/2} e^{-j\omega})^n$$

$$= \frac{1}{1 - a^{1/2} e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - \sqrt{a}}$$

∴ This shows that spectrum in part (ii) is simply the fourier transform of $x(n/2)$.

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## Sampling rate Conversion by a rational factor '$\frac{I}{D}$' :-

In some applications, sampling rate conversion by a non-integer factor may be required. For example, transfering data from a compact disc at a rate of 44.1 KHz to a digital audio tape at 48KHz. Hence, the sampling rate conversion factor is 48/44.1, which is a non-integer.

A sampling rate conversion by a factor '$\frac{I}{D}$' can be achieved by first performing interpolation by factor 'I' and then performing decimation by a factor 'D'. The sampling rate conversion can be accomplished by cascading an interpolator with a decimator as shown in fig.

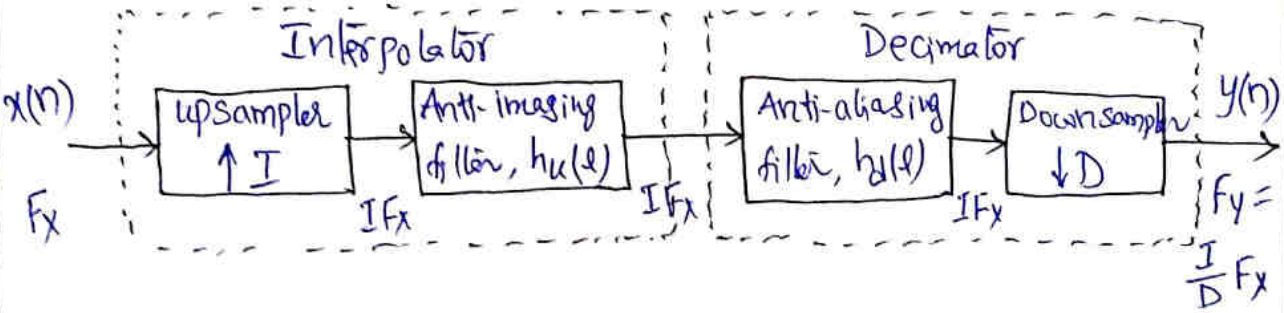



Fig:- Block digram of a method for sampling rate convertion.



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From fig, the two filters $h_u(l)$ & $h_d(l)$ are operated at the same sampling rate and both hence combined into a single filter with impulse response $h(l)$.

The frequency response of anti-imaging filter $h_u(l)$ is given as

$$H_u(\omega) = \begin{cases} I & ; \quad \frac{-\pi}{I} \le \omega \le \frac{\pi}{I} \\ 0 & ; \quad \text{otherwise} \end{cases} \longrightarrow (1)$$

The frequency response of anti-aliasing filter ~~if~~ $h_d(l)$ is given as

$$H_d(\omega) = \begin{cases} 1 & ; \quad \frac{-\pi}{D} \le \omega \le \frac{\pi}{D} \\ 0 & ; \quad \text{otherwise} \end{cases} \longrightarrow (2)$$

The overall cascading effect of LPF's will have a cut off frequency which is minimum of $\frac{\pi}{I}$ & $\frac{\pi}{D}$.

Hence, the frequency response of combined filter is

$$H(\omega) = \begin{cases} I & ; \quad |\omega| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & ; \quad \text{otherwise} \end{cases} \longrightarrow (3)$$

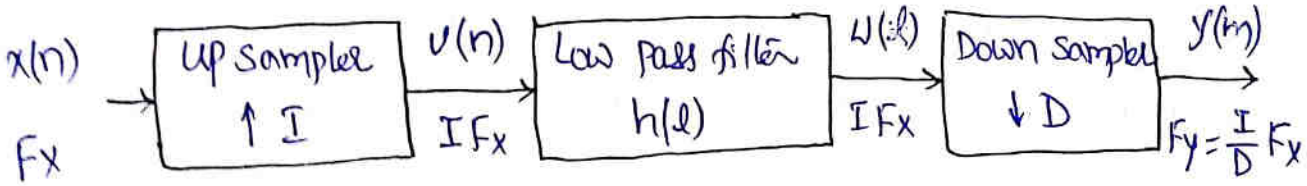



Fig:- Sampling rate conversion by a factor $\frac{I}{D}$


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Derivation for output $y(n)$ :-

From fig, $v(n) = x(n/I)$ ⟶ ① $[\because x(n) = v(nI)]$

$$w(l) = v(n) * h(l)$$

$$= \sum_{k=-\infty}^{\infty} h(l-k).v(k) = \sum_{k=-\infty}^{\infty} h(l-kI).v(kI)$$

[∵ from interpolation]

$$\Rightarrow w(l) = \sum_{k=-\infty}^{\infty} h(l-kI).x(k) \longrightarrow ②$$

[∵ from ①]

$$y(m) = w(mD)$$

$$\Rightarrow y(m) = \sum_{k=-\infty}^{\infty} h(mD-kI).x(k) \longrightarrow ③$$

This is the equation for output sequence.

Relation between spectrums $X(\omega)$ & $Y(\omega)$ :-

→ From Fig, the output of I up sampler is

$$V(\omega_v) = X(\omega_v I) \longrightarrow ④$$

→ The signal $v(n)$ is passed through a filter $h(l)$ and the output signal is $w(l)$. Hence,

$$W(\omega_v) = \begin{cases} I.X(\omega_v I) & ; |\omega| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & ; \text{otherwise} \end{cases}$$

⟶ ⑤

→ From decimator, W.K.T, $Y(\omega_y) = \frac{1}{D}.X\left(\frac{\omega_y}{D}\right)$ ⟶ ⑥

In fig, input is $w(l)$. So, eqn ⑥ written as

$$Y(\omega_y) = \frac{1}{D}.W\left(\frac{\omega_y}{D}\right) \longrightarrow ⑦$$

We have, $\omega_y = D \cdot \omega_v$ ⟶ (8)

Hence, $Y(\omega_y) = \frac{1}{D} \cdot H(\omega_0)$ ⟶ (9)

from (5) & (9).

$$Y(\omega_y) = \begin{cases} \frac{I}{D} \cdot X(\omega_0 I) & : |\omega_0| \leq \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & : \text{otherwise} \end{cases}$$

from (8).

$$Y(\omega_y) = \begin{cases} \frac{I}{D} \cdot X\left(\frac{I}{D}, \omega_y\right) & : \left|\frac{\omega_y}{D}\right| \leq \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & ; \text{otherwise} \end{cases}$$

⟶ (10)

It can be written as

$$Y(\omega_y) = \begin{cases} \frac{I}{D} \cdot X\left(\frac{I}{D}\omega_y\right) & ; |\omega_y| \leq \min\left(\pi, \frac{D}{I}\pi\right) \\ 0 & : \text{otherwise} \end{cases}$$ ⟶ (11)

∴ This is the required equation that relates the spectrums of input and output.

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## Sampling rate Conversion of band pass signals:-

**Principle** - The band pass signal is converted to low pass signal with the help of mixing. The low pass signal contains all the information that is present in the band pass signal. The low pass signal is then interpolated or decimated.

**Block diagram & Operation:-**

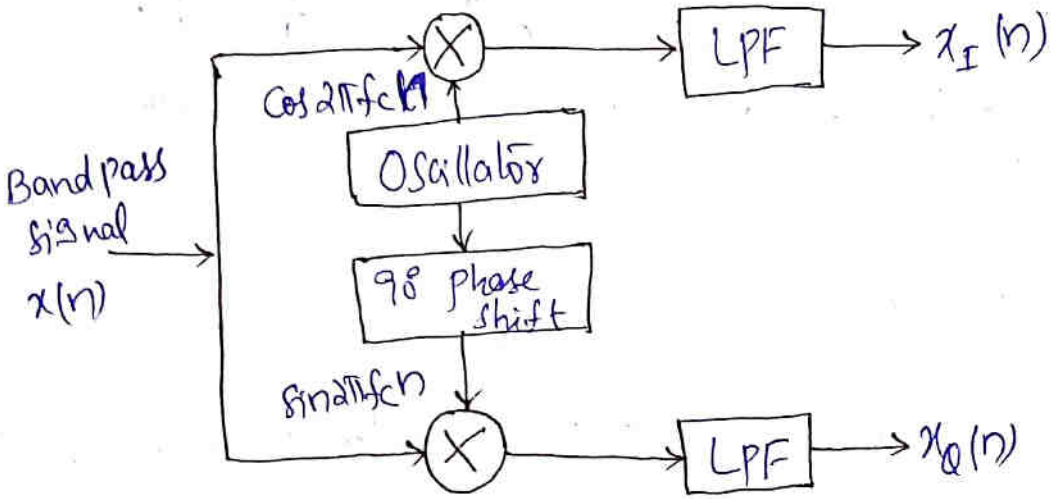



Fig:- Frequency Translator.

The input band pass signal $x(n)$ can be represented in its inphase and quadrature components, such as

$$x(n) = x_I(n).\cos(2\pi f_c n) - x_Q(n).\sin(2\pi f_c n) \longrightarrow (1)$$

Here $x_I(n)$ is inphase component and $x_Q(n)$ is quadrature component. $f_c$ is the centre frequency of the band pass signal.

The oscillator generates quadrature signals $\cos(2\pi f_c n)$ and $\sin(2\pi f_c n)$. These signals are mixed with input



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band pass signal $x(n)$. The lowpass filter preserves only difference frequency components. These difference frequency components are nothing but quadrature components $x_I(n)$ & $x_Q(n)$.

## Decimation of band pass signals:-

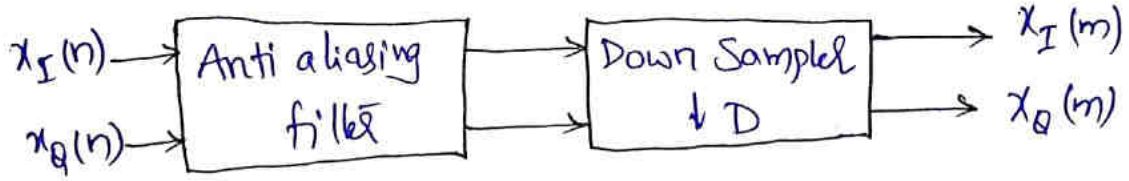



Fig:- Decimator for band pass signals.

The response of Combined filter nothing but anti-aliasing filter of decimator and low pass filter of frequency translator is given as

$$H(\omega_x) = \begin{cases} 1 & : \text{for } |\omega_x| \leq \pi/D \\ 0 & : \text{otherwise} \end{cases}$$

Here '$\omega_x$' is any desired frequency in the range, $0 \leq \omega_x \leq \pi$

## Interpolation of band pass signals:-

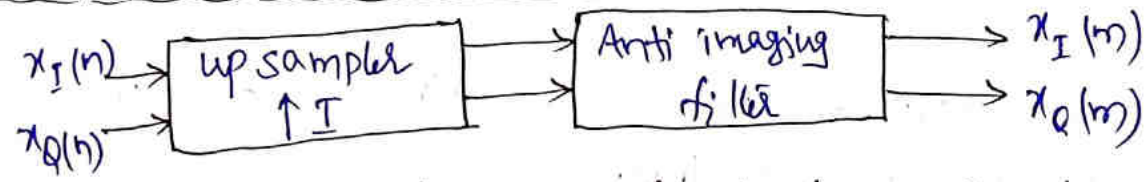



Fig:- Interpolator for band pass signals.

The frequency response is given as

$$H(\omega_y) = \begin{cases} I & \text{for } |\omega_y| \leq \frac{\omega}{2I} \\ 0 & : \text{otherwise} \end{cases}$$

Here '$\omega_y$' is the frequency at the output of upsampler.

The sampling rate conversion $I/D$ can be achieved by cascade of interpolator and decimator on the inphase and quadrature components of band pass signals.

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Eg: Show that the up-sampler and down-sampler are time variant systems.

Sol:-

i). Consider a factor of 'L' up.sampler is given by

$$y(n) = x(n/L)$$

- Input delayed by 'K' samples. $y(n,K) = x\left(\frac{n}{L} - K\right)$
- Output delayed by 'K' samples. $y(n-K) = x\left(\frac{n-K}{L}\right)$

$\therefore y(n,K) \neq y(n-K)$, therefore the upsampler is a time variant system

ii). Consider a factor of 'M' down sampler is given by

$$y(n) = x(Mn)$$

- Input delayed by 'K' samples, $y(n,K) = x(Mn-K)$
- Output delayed by 'K' samples, $y(n-K) = x(M(n-K))$

$\therefore y(n,K) \neq y(n-K)$, therefore the down sampler is a time variant system.

= 0 =

Formula for designing / implementing sampling rate Conversions:

* Normalized transition bandwidth, $\Delta f = \frac{f_s - f_p}{f_s'}$

* Length of filter 'N', $N = \frac{-20 \cdot \log_{10}\sqrt{\delta_p \delta_s} - 13}{14.6\,\Delta f} + 1$

* the variance of total round off error is given by

$$\sigma^2 = \frac{(N-1)\,2^{-2q}}{12}$$



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Eg:- Design a decimator with the following specification: $D=5$, $\delta_p = 0.025$, $\delta_s = 0.0035$, $f_s = 100Hz$. Assume any other required data.

Sol:- Given that, for a decimator, $D = 5$

Pass band ripple, $\delta_p = 0.025$,

stop band ripple, $\delta_s = 0.0035$ & $f_s = 100Hz$.

Assume: Pass band frequency, $f_p = 80Hz$ &

sampling frequency, $f_s' = 20KHz$.

So, the normalized transition band is given by

$$\Delta f = \frac{f_s - f_p}{f_s'} = \frac{100-80}{20,000} = 0.001Hz$$

The order of the filter, $N = \frac{-20 \cdot \log_{10}\sqrt{\delta_p \delta_s} - 13}{14.6\,\Delta f} + 1$.

$$\Rightarrow N = \frac{-20 \cdot \log_{10}\sqrt{0.025 \times 0.0035} - 13}{14.6 \times 0.001} + 1 = 1890.03 \approx 1891$$

The output frequency is $f_y = \frac{f_x}{D} = \frac{f_s'}{D} = \frac{20K}{5} = 4KHz$.

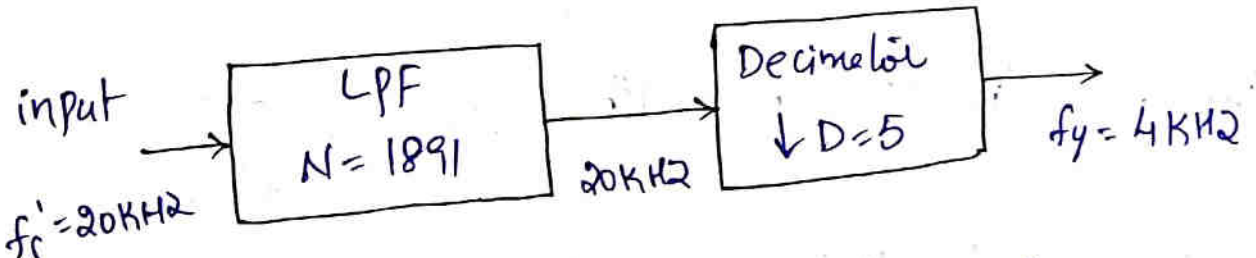



Fig:- Single stage decimation by a factor of 5.

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Eg:- A one stage decimator is characterized by the following decimator factor = 3. Anti aliasing filter coefficients $h(0) = -0.06 = h(4)$, $h(1) = 0.3 = h(3)$, $h(2) = 0.62$. Given the data $s(n)$ with successive values $[6, -2, -3, 8, 6, 4, -2]$. Calculate and list the filtered output and the output of the decimator.

Sol:- Given that, one stage decimator with factor, $D = 3$

Anti aliasing filter coefficients are.

$h(0) = -0.06$, $h(1) = 0.3$, $h(2) = 0.62$, $h(3) = 0.3$, $h(4) = -0.06$

$s(n) = [6, -2, -3, 8, 6, 4, -2]$

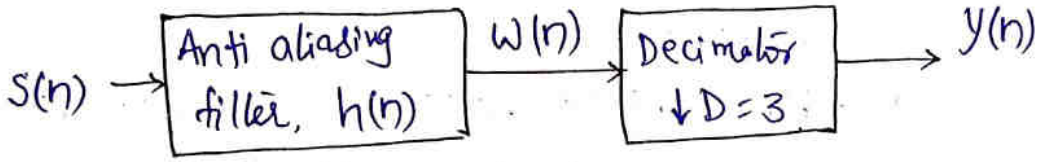



Fig:- Decimation process with factor D

i). Filtered output, $w(n)$ :-

$$w(n) = h(n) * s(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot s(n-k)$$

$$w(0) = \sum_{k=-\infty}^{\infty} h(k) \cdot s(-k) = h(0) \cdot s(0) = -0.36$$

$$w(1) = \sum_{k=-\infty}^{\infty} h(k) \cdot s(1-k) = h(0) \cdot s(1) + h(1) \cdot s(0) = 1.92$$

$$w(2) = \sum_{k=-\infty}^{\infty} h(k) s(2-k) = h(0)s(2) + h(1)s(1) + h(2)s(0) = 3.3$$

$$w(3) = \sum_{k=-\infty}^{\infty} h(k)\, s(3-k) = h(0)s(3) + h(1)s(2) + h(2)s(1) + h(3)s(0) = -0.82$$

$$w(4) = \sum_{k=-\infty}^{\infty} h(k) s(4-k) = h(0)s(4) + h(1)s(3) + h(2)s(2) + h(3)s(1) + h(4)s(0)$$

$$= -0.78$$



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$$w(5) = \sum_{k=-2}^{\infty} h(k)\, s(5-k) = h(0)s(5) + h(1)s(4) + h(2)s(3) + h(3)s(2) + h(4)s(1) + h(5)s(0)$$

$$= -5.74$$

$$w(6) = \sum_{k=-2}^{\infty} h(k)\, s(6-k) = h(0)s(6) + h(1)s(5) + h(2)s(4) + h(3)s(3) + h(4)s(2) + h(5)s(1) + h(6)s(0)$$

$$= 7.62$$

$$w(7) = \sum_{k=-2}^{\infty} h(k)\, s(7-k) = h(0)s(7) + h(1)s(6) + h(2)s(5) + h(3)s(4) + h(4)s(3) + h(5)s(2) + h(6)s(1) + h(7)s(0)$$

$$= 3.2$$

$$w(8) = \sum_{k=-2}^{\infty} h(k)\, s(8-k) = h(0)s(8) + h(1)s(7) + h(2)s(6) + h(3)s(5) + h(4)s(4) + h(5)s(3) + h(6)s(2) + h(7)s(1) + h(8)s(0)$$

$$= -0.4$$

$$w(9) = \sum_{k=-2}^{\infty} h(k)\, s(9-k) = h(0)s(9) + h(1)s(8) + h(2)s(7) + h(3)s(6) + h(4)s(5) + h(5)s(4) + h(6)s(3) + h(7)s(2) + h(8)s(1) + h(9)s(0)$$

$$= -0.84$$

$$w(10) = \sum_{k=-2}^{\infty} h(k)\, s(10-k) = h(0)s(10) + h(1)s(9) + h(2)s(8) + h(3)s(7) + h(4)s(6) + h(5)s(5) + h(6)s(4) + h(7)s(3) + h(8)s(2) + h(9)s(1) + h(10)s(0)$$

$$= 0.12$$

$$w(n) = [-0.36,\ 1.92,\ 3.3,\ -0.82,\ -0.78,\ 5.74,\ 7.62,\ 3.2,\ -0.4,\ -0.84,\ 0.12]$$

ii. Decimator output:-

$$y(n) = w(nD) = w(3n)$$

$$\therefore y(n) = [-0.36,\ -0.82,\ 7.62,\ -0.84]$$

$$= 0$$


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Eg:- The sequence $x(n) = \{0,2,4,6,8\}$ is interpolated using interpolation ~~using~~ sequence $b_k = \{1/2, \underset{\uparrow}{1}, 1/2\}$ and the interpolation factor is 2. Find the interpolated sequence $y(n)$.

Sol:- Given that, $x(n) = \{0,2,4,6,8\}$ & $I = 2$

interpolation sequence - $b_k = \{1/2, \underset{\uparrow}{1}, 1/2\}$

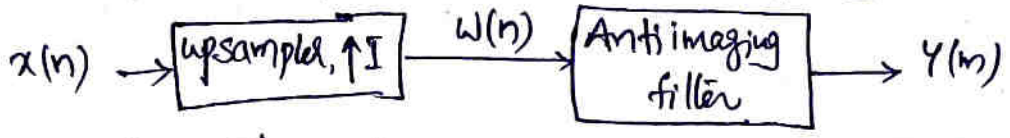



Fig:- Block diagram of interpolation by a factor of 'I'

$w(n) = \{0,0,2,0,4,0,6,0,8\}$

The interpolated sequence is

$$y(n) = \sum_{k=-\alpha}^{\alpha} b_k \cdot w(n-k)$$

$$\Rightarrow y(n) = b_{-1} w(n+1) + b_0 w(n) + b_1 \cdot w(n-1)$$
$$= 0.5w(n+1) + w(n) + 0.5 w(n-1)$$

If $n=0$, $y(0) = 0.5w(1) + w(0) + 0.5w(-1) = 0$

$n=1$, $y(1) = 0.5w(2) + w(1) + 0.5\{w(0) = 1$

$n=2$, $y(2) = 0.5w(3) + w(2) + 0.5w(1) = 2$

$n=3$, $y(3) = 0.5w(4) + w(3) + 0.5w(2) = 3$

$n=4$, $y(4) = 0.5w(5) + w(4) + 0.5w(3) = 4$

$n=5$, $y(5) = 0.5w(6) + w(5) + 0.5w(4) = 5$

$n=6$, $y(6) = 0.5w(7) + w(6) + 0.5w(5) = 6$

$n=7$, $y(7) = 0.5w(8) + w(7) + 0.5w(6) = 7$

$n=8$, $y(8) = 0.5w(9) + w(8) + 0.5w(7) = 8$

$\therefore y(n) = \{0,1,2,3,4,5,6,7,8\}$

$= 0 =$



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Eg:- A signal $x(n)$ at a sampling frequency of 2.048 KHz is to be decimated by a factor of 32 to yield a signal at a sampling frequency of 64 Hz. The signal band of interest extends from 0 to 30 Hz. The anti aliasing digital filter should satisfy the following specifications:

Pass band deviation : 0.01 dB  stop band deviation : 80 dB

Pass band : 0-30 Hz  stop band : 32-64 Hz

The signal components in the range from 30 to 32 Hz should be protected from aliasing. Design a suitable two stage decimator.

Sol:- Given that,

input sampling frequency, $f_s' = 2.048$ KHz

Decimation factor, $D = 32$

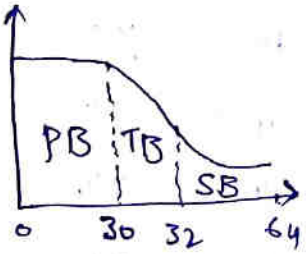
output sampling frequency, $f_o = 64$ Hz

Pass band deviation $= 0.01$ dB ; stop band deviation $= 80$ dB

$-20 \cdot \log_{10} \delta_p = 0.01$ ; $-20 \cdot \log_{10} \delta_s = 80$

$\Rightarrow \delta_p = 0.9988$ ; $\Rightarrow \delta_s = 0.0001$

$\therefore f_p = 30$ Hz & $f_s = 32$ Hz



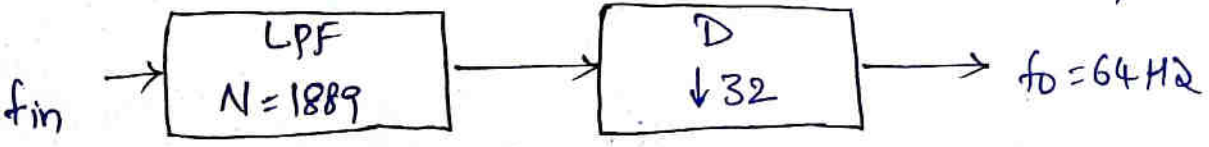

For single stage decimator:-

The normalized transition bandwidth, $\Delta f = \frac{f_s - f_p}{f_s'}$

$\Rightarrow \Delta f = \frac{32-30}{2048 \times 10^3} = 9.766 \times 10^{-4}$

filter length, $N = \frac{-20 \cdot \log_{10} \sqrt{\delta_p \delta_s} - 13}{14.6 \Delta f} + 1 = 1888.46$

$\approx 1889$





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For Two stage decimator:-

Consider two decimator factor as $D_1 = 2$ & $D_2 = 16$

The output sampling frequencies are

$f_1 = \frac{2.048\,KHz}{2} = 1.024\,KHz$ & $f_2 = \frac{2.048\,KHz}{16} = 0.18KHz$

a). For first stage:

Pass band : 0 to 30 Hz

Transition band : $30Hz \le f \le f_1 - f_s$

$= 30 \le f \le 1.024\,KHz - 32$

$= 30 \le f \le 992$

The normalized transition bandwidth, $\Delta f_1 = \frac{f_s - f_p}{f_s'}$

$\therefore \Delta f_1 = \frac{992 - 30}{2.048 \times 10^3} = 0.4697$

$\delta_p' = \frac{\delta_p}{2} = \frac{0.9988}{2} = 0.4994$

Order of filter, $N_1 = \frac{-20.\log\sqrt{\delta_p' \delta_s} - 13}{14.6\,\Delta f_1}$

$\simeq 4.47 \simeq 5$

b). For second stage:-

Pass band : $0 \le f \le 30$

Transition band : $30 \le f \le f_2 - f_s$

$= 30 \le f \le 0.128 \times 10^3 - 32$

$= 30 \le f \le 96$

$\therefore \Delta f_2 = \frac{96 - 30}{2.048 \times 10^3} = 0.0322$



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$$\delta_s' = \frac{\delta_s}{2} = \frac{0.0001}{2} = 5 \times 10^{-5}$$

The order of filter, $N_2 = \dfrac{-20.\log_{10}\sqrt{\delta_p \delta_s'} - 13}{14.6\,\Delta f_2}$

$$\Rightarrow N_2 = \frac{-20.\log_{10}\sqrt{0.9988 \times 5 \times 10^{-5}} - 13}{14.6 \times 0.0322}$$

$$\Rightarrow N_2 = 63.85 \approx 64$$

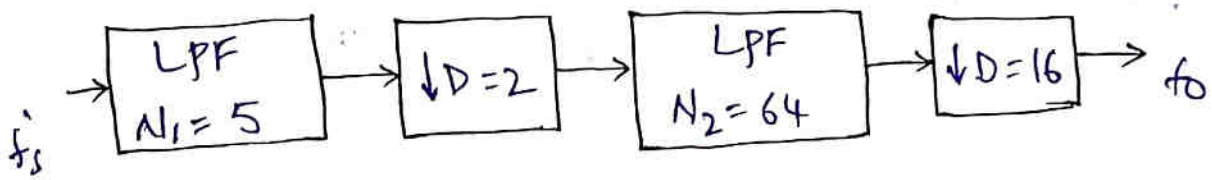



Fig:- Two stage decimator

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## Need for Multirate signal processing:-

- They possess reduced computations
- They require less storage space
- Filters used in multirate system has simple design problems
- They have less finite word length effects.

## Drawbacks in multistage implementation:-

→ The complexity of the multirate DSP increases with the inclusion of pre and post processing devices like ADC & DAC's and associated reconstruction filters.

→ These systems also suffer from frequency limitations

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# Applications of Multirate DSP:-

## ①. Subband Coding of Speech Signals:-

**Necessity:-** Speech energy is not uniformly distributed over the complete frequency range. More energy is contained in lower frequency bands and less energy is contained in higher frequency bands. Hence lower frequency bands are coded with more number of bits and high frequency bands are coded with less number of bits.

**Definition:-** The speech signal is divided into several frequency bands. These frequency bands do not overlap each other. They are called subbands. Each subband is encoded seperately.

**Block diagram:-**

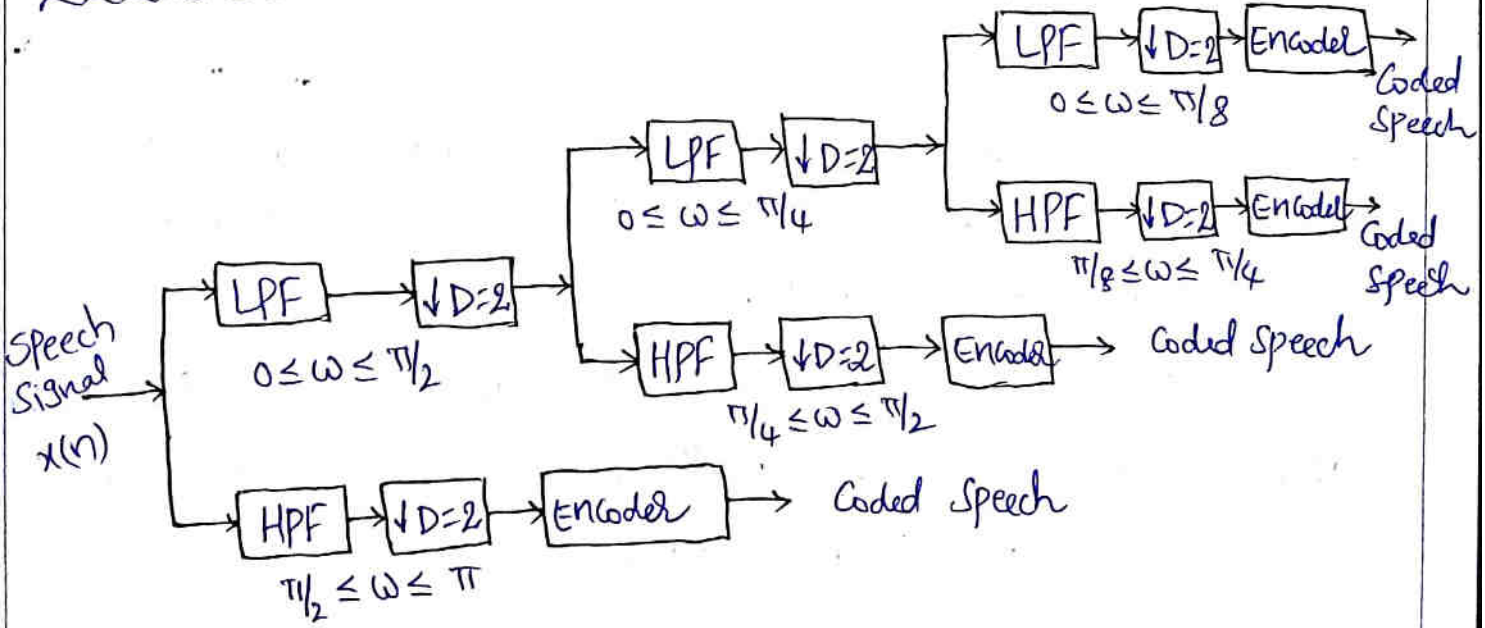
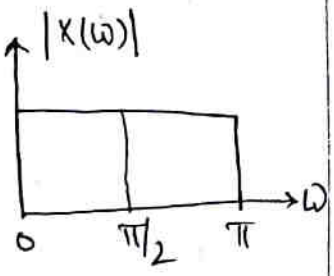



Fig:- Block diagram of subband coding.

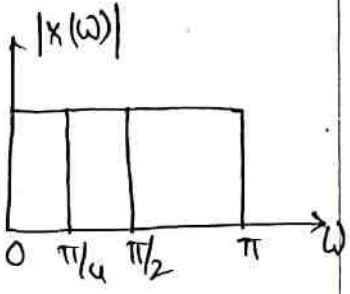

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→ The speech signal containing the frequencies from '0' to 'π' radians is passed through a pair of LPF & HPF.

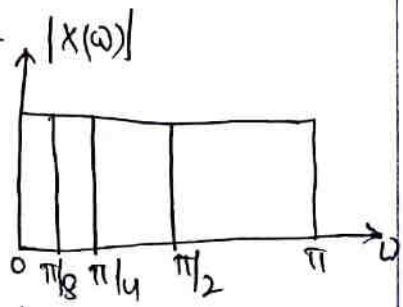
→ These LPF's & HPF's split the signal into non-overlapping frequency bands. These bands are $0 \le \omega \le \pi/2$ and fig. shows these bands.




→ The lower frequency band is further split into frequency bands i.e., $0 \le \omega \le \pi/4$ and $\frac{\pi}{4} \le \omega \le \pi/2$. This is shown in fig. Note that higher frequency band of $\frac{\pi}{2} \le \omega \le \pi$ is not split further. It is directly encoded since it contains less energy.




→ The lowest frequencies band is further split into $0 \le \omega \le \pi/8$ & $\pi/8 \le \omega \le \pi/4$. Thus the signal is split into four frequency bands as shown in fig.




→ The subbands are decimated immediately after they are ~~filtered~~ filtering. Hence the sampling frequency can also be reduced.

From fig, observe that decimation is performated after every stage of splitting. This requires less number of bits for encoding.


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## Fillers used for Subband Coding:-

Fig. shows the characte-ristic of LPF & HPF used for subband coding. It is the characteristics of Quadrature Mirror Filter (QMF). These are physically realizable filters. They eliminate the aliasing resulting due to decimation.

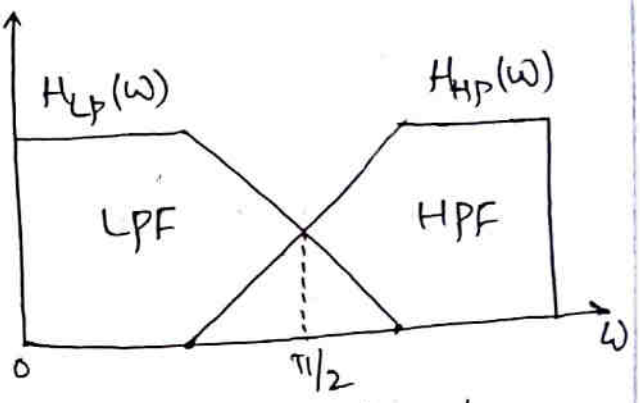



Fig:- QMF characteristics

### Applications of Subband Coding:-

i). Coding and Compression of speech signals.

ii). Data compression of image signals.

iii). It provides bandwidth compression, when the energy of the signal is unevenly spread over the spectrum.

### Synthesis of subband coded signal:-

Fig. shows the block diagram of synthesis of subband coded signal.

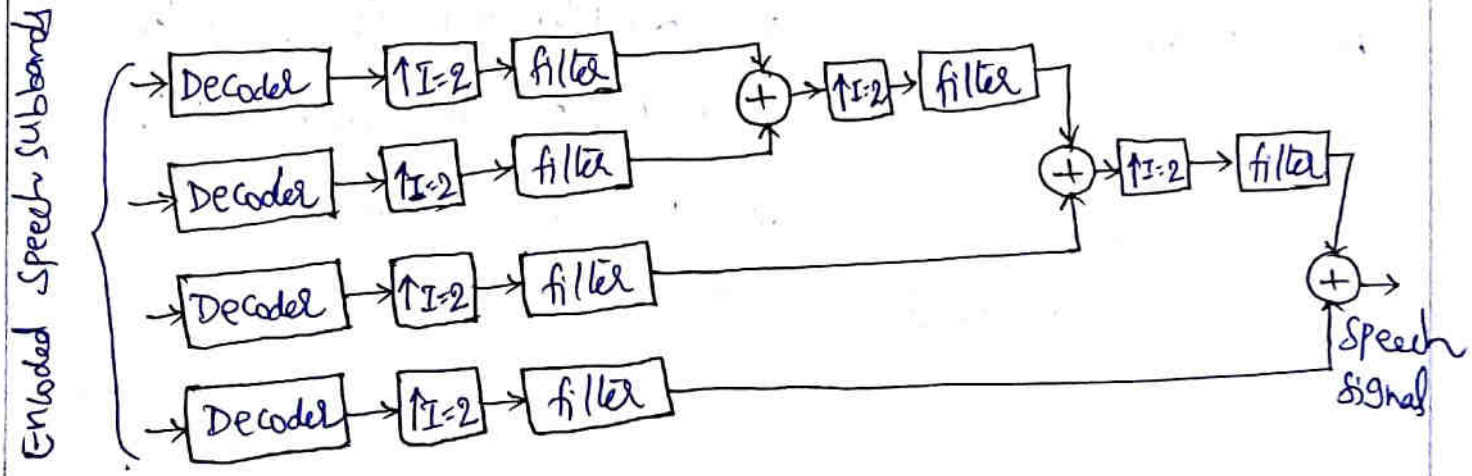



Fig: Block diagram of synthesis of subband coded signal




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Observe that, the above diagram performs exactly the reverse function of Analysis operation.

* Interpolation of two subband makes their sampling rates equal. Then the two signals are filtered and combined.

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## Digital filter banks:-

Filter banks are of two types, Analysis filter bank and synthesis filter bank.

### i). Analysis filter bank:-

The M-channel analysis filter bank is shown in fig. It consists of M-sub filters. The individual subfilter $H_k(z)$ is known as analysis filter. All the subfilters are equally spaced in frequency and each have the same bandwidth. The spectrum of input signal $X(e^{j\omega})$ lies in the range $0 \le \omega \le \pi$. The filter bank splits the signal into number of subbands each having a bandwidth of $\frac{\pi}{M}$. The filter $H_0(z)$ is low pass, $H_1(z)$ to $H_{M-2}(z)$ are band pass and $H_{M-1}(z)$ is high pass. As spectrum of signal is band limited to $\frac{\pi}{M}$, the sampling rate can be reduced by a factor 'M'. The down sampling moves all the subband signals into the base band $0 \le \omega \le \pi/M$.


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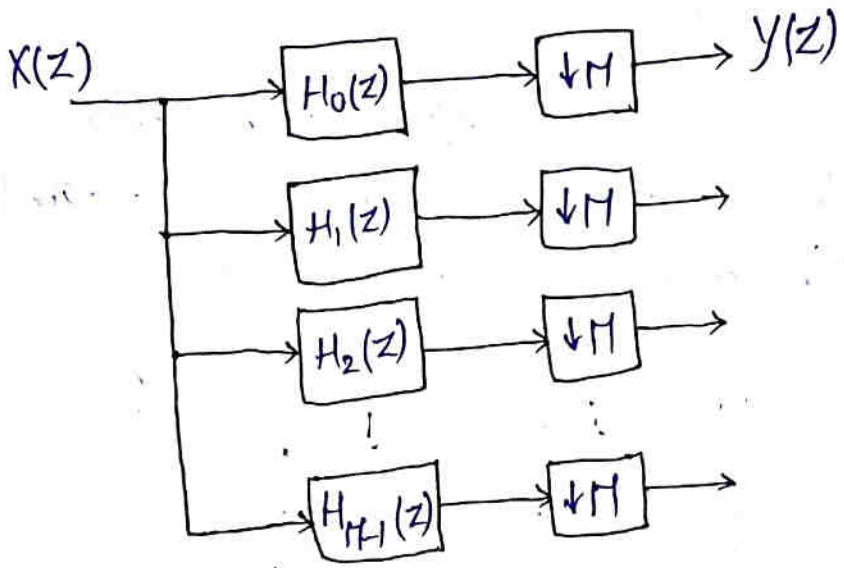



Fig:- Analysis filter bank

ii). Synthesis filter bank:-

The M-channel synthesis filter bank is dual of M-channel analysis filter bank. In this case, each $U_m(z)$ is fed to an upsampler. The upsampling process produces the signal $U_M(z^M)$. These signals are applied to filters $G_m(z)$ and finally added to get the output signal $\hat{X}(z)$. The filter $G_0(z)$ to $G_{M-1}(z)$ have the same characteristics as the analysis filters $H_0(z)$ to $H_{M-1}(z)$.

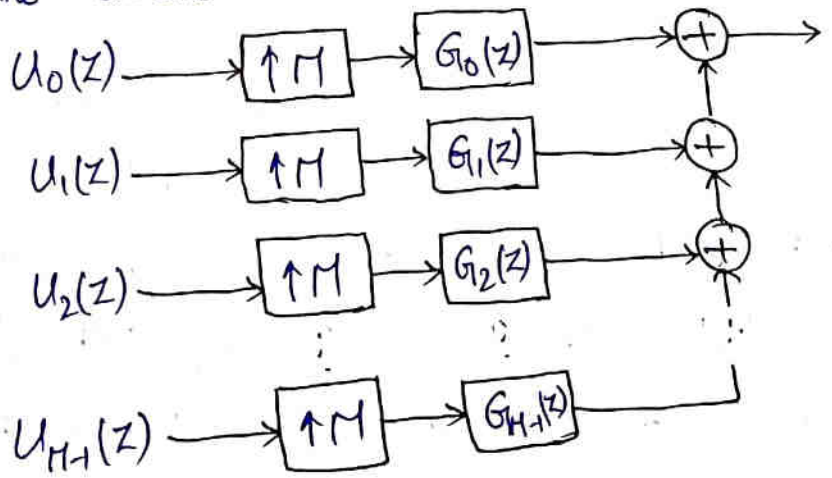



Fig:- Synthesis filter bank.

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iii). Subband Coding filter bank:-

If we combine the analysis filter bank and synthesis filter bank, we obtain an M-channel subband coding filter bank as shown in fig.

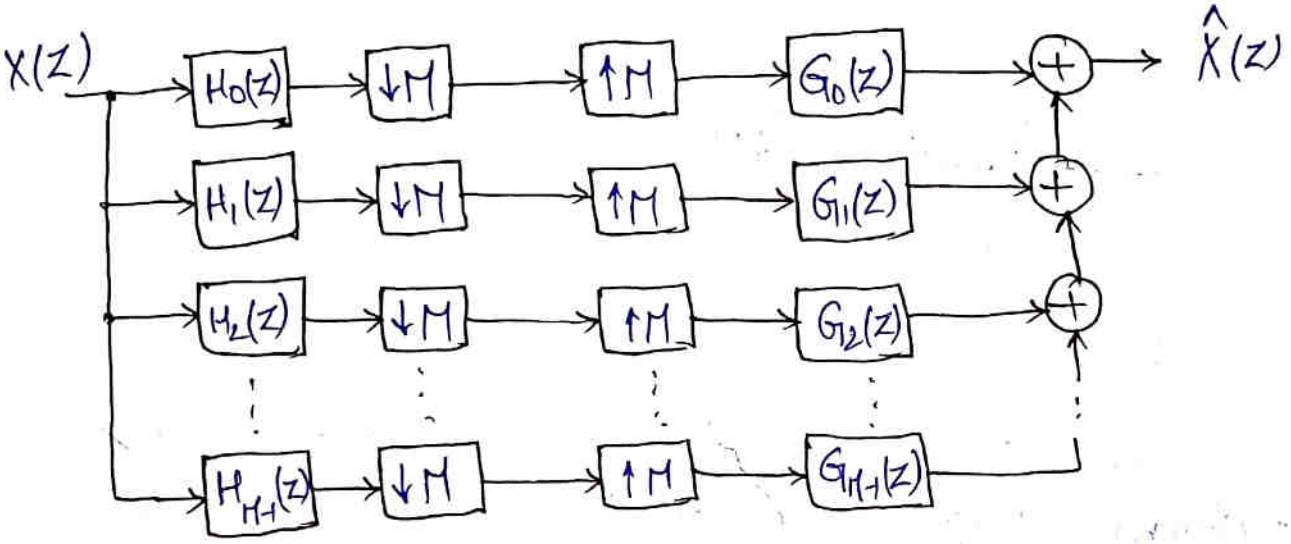



Fig:- Subband Coding filter bank

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Transmultiplexers:-

The transmultiplexer is multi-input, multi-output multirate structure. It translates the TDM signals to FDM format and vice versa.

Fig. shows the block diagram of basic L-channel transmultiplexer. It consists of L-channel synthesis filter bank at the input end followed by L-channel analysis filter bank at the output end. Note that any path of synthesis filter bank can be connected to any path of the analysis filter bank.


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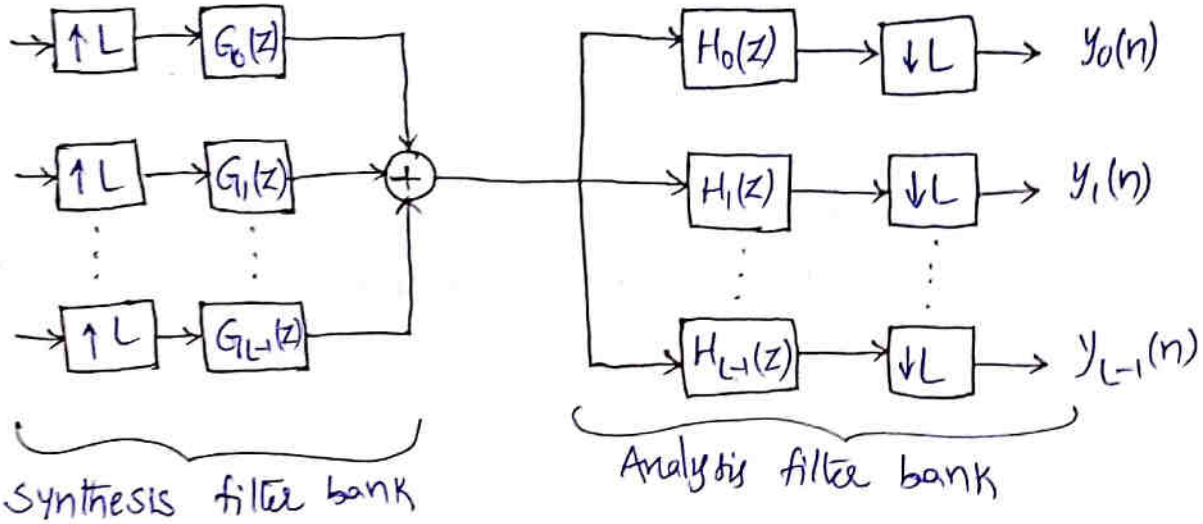



Fig:- L- channel transmultiplexer structure

Let the $l^{th}$ path of synthesis filter bank is connected to $k^{th}$ path of analysis filter bank. Fig. shows this connection. The i/p - o/p relation of the transmultiplexer is described in such connection as,

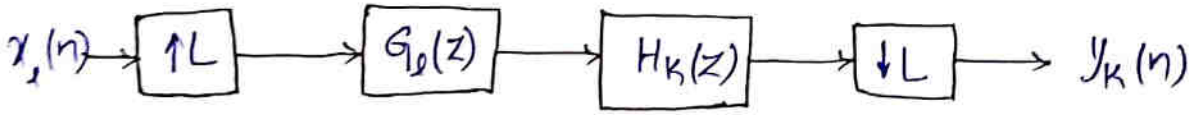



Fig:- $k-l^{th}$ path of transmultiplexer operation

$$Y_k(z) = \sum_{l=0}^{L-1} G_l(z) \cdot H_k(z) \cdot X_l(z) \longrightarrow (1)$$

$$= \sum_{l=0}^{L-1} F_{kl}(z) \cdot X_l(z) \longrightarrow (2)$$

Here $F_{kl}(z) = G_l(z) \cdot H_k(z)$ is $L \times L$ size matrix.

Here $Y_k(z)$ is $L \times 1$ size o/p matrix and $X_l(z)$ is $L \times 1$ size input matrix.

Here $$F_{kl}(z) = \begin{cases} 0 & ; \text{for } k \neq l \\ \alpha_k z^{-n_k} & ; \text{for } k = l \end{cases} \longrightarrow (3)$$

(d)


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From ② & ③,

$$Y_k(z) = F_k(z) \cdot X_k(z) = \alpha_k \cdot z^{-n_k} \cdot X_k(z)$$

$$\Rightarrow y_k(n) = \alpha_k \cdot x(n - n_k) \longrightarrow ④$$

Thus, the output is perfect replica of the input.

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