UNIT-V: DYNAMIC PROGRAMMING

When formulating LP's we often found that, strictly, certain variables should have been regarded as taking integer values but, for the sake of convenience, we let them take fractional values reasoning that the variables were likely to be so large that any fractional part could be neglected. While this is acceptable in some situations, in many cases it is not, and in such cases we must find a numeric solution in which the variables take integer values.

Problems in which this is the case are called integer programs (IP's) and the subject of solving such programs is called integer programming (also referred to by the initials IP).

IP's occur frequently because many decisions are essentially discrete (such as yes/no, do/do not) in that one or more options must be chosen from a finite set of alternatives.

An IP in which all variables are required to be integers is called a pure IP problem.

If some variables are restricted to be integer and some are not then the problem is a mixed IP problem.

The case where the integer variables are restricted to be 0 or 1 comes up surprising often. Such problems are called pure (mixed) 0-1 programming problems or pure (mixed) binary IP problems.

For any IP we can generate an LP by taking the same objective function and same constraints but with the requirement that variables are integer replaced by appropriate continuous constraints:

"$x_i \geq 0$ and integer" can be replaced by $x_i \geq 0$

"$x_i = 0$ or 1" can be replaced by $x_i \geq 0$ and $x_i \leq 1$

The LP obtained by omitting all integer or 0-1 constraints on variables is called LP Relaxation of the IP (LR).

FORMULATING IP

Practical problems can be formulated as IPs. For instance budgeting problems, knapsack problems, fixed charge production and location problems, set covering problems, etc.

Budgeting Problems Example

CapitalBudgeting

(Winston 9.2, p. 478 – modified)

Stock is considering four investments

Each investment yields a determined NPV ($8,000, $11,000, $6,000, $4,000) Each investment requires at certain cash flow at the present time ($5,000, $7,000,

$4,000, $3,000)

Currently Stock has $14,000 available for investment.

1 www.jntufastupdates.com

Formulate an IP whose solution will tell Stock how to maximize the NPV obtained from the four investments.

Answer

Begin by defining a variable for each decision that Stockco must make. In this case, we will use a 0-1 variable xj for each investment:

If xj is 1 then Stock will make investment j. If it is 0, Stock will not make the investment. This leads to the 0-1 programming problem:

max z = 8x1 + 11x2 + 6x3 + 4x4

s.t.5x1 + 7x2 + 4x3 + 3x4 ≤ 14

xj = 0 or 1 (j = 1,2,3,4)

Comment

Now, a straightforward "bang for buck" (taking ratios of objective coefficient over constraint coefficient) suggests that investment 1 is the best choice.

Ignoring integrality constraints, the optimal linear programming solution is:

x1 = x2 = 1, x3 = 0.5, and x4 = 0 for a value of $22K Unfortunately, this solution is not integral. Rounding x3 down to 0:

x1 = x2 = 1, x3 = x4 = 0 for a value of $19K

There is a better integer solution (optimal solution):

x1 = 0, x2 = x3 = x4 = 1 for a value of $21K

This example shows that rounding does not necessarily give an optimal value.

Example 1.b. Multiperiod

There are four possible projects, which each run for three years and have the following characteristics: Which projects would you choose in order to maximize the total return?

Capital requirements

Answer

We will use a 0-1 variable xj for each project:

xj is 1 if we decide to do project j;

xj is 0 otherwise (i.e. not do project j). This leads to the 0-1 programming problem: max 0.2 x1 + 0.3 x2 + 0.5 x3 + 0.1 x4

s.t.0.5 x1 + 1x2 + 1.5 x3 + 0.1 x4 ≤ 3.1

0.3 x1 + 0.5 x2 + 1.5 x3 + 0.4 x4 ≤ 2.5

0.2 x1 + 0.2 x2 + 0.3 x3 + 0.1 x4 ≤ 0.4

xj = 0 or 1 j = 1, … 4

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Example 1.c. Capital Budgeting Extension

There are a number of additional constraints Stock might want to add. Logical restrictions can be enforced using 0-1 variables:

Stock can only make two investments

$x1 + x2 + x3 + x4 \leq 2$

Any choice of three or four investments will have $x1 + x2 + x3 + x4 \geq 3$ If investment 2 is made, investment 4 must also be made

$x2 < x4$ or $x2 – x4 \leq 0$

If x2 is 1, then x4 is also 1 as Stock desires; if x2 is 0, then there is no restriction for x4 (x4 is 0 or 1)

If investment 1 is made, investment 3 cannot be made

$x1 + x3 \leq 1$

If x1 is 1, then x3 is 0 as Stock desires; if x1 is 0, then there is no restriction for

x3 (x3 is 0 or 1)

Either investment 1 or investment 2 must be done

$x1 + x2 = 1$

If x1 is 1, then x2 is 0 (only investment 1 is done); if x1 is 0, then x2 is 1 (only investment 2 is done)

Knapsack Problems

Any IP that has only one constraint is referred to as a knapsack problem. Furthermore, the coefficients of this constraint and the objective are all non-negative. The traditional story is that: There is a knapsack. There are a number of items, each with a size and a value. The objective is to maximize the total value of the items in the knapsack.

Knapsack problems are nice because they are (usually) easy to solve.

Example 2. Knapsack

For instance, the following is a knapsack problem:

Maximize$8 x1 + 11 x2 + 6 x3 + 4 x4$

Subject to$5 x1 + 7 x2 + 4 x3 + 3 x4 \leq 14$

$xj = 0$ or $1j = 1, \ldots 4$

Fixed Charge Problems

There is a cost associated with performing an activity at a nonzero level that does not depend on the level of the activity.

An important trick can be used to formulate many production and location problems involving the idea of a fixed charge as IP.

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Example 3.a. Gandhi

(Winston 9.2, p. 480)

Gandhi Co makes shirts, shorts, and pants using the limited labor and cloth described below.

In addition, the machinery to make each product must be rented.

Shirts Shorts Pants Total Avail.

Answer

Let xj be number of clothing produced.

Let yj be 1 if any clothing j is manufactured and 0 otherwise.

Profit = Sales revenue – Variable Cost – Costs of renting machinery For example the profit from shirts is

$z1 = (12 – 6) x1 – 200 y1$

Since supply of labor and cloth is limited, Gandhi faces two constraints.

To ensure $xj > 0$ forces $yj = 1$, we include the additional constraints

$xj \leq Mj\ yj$

From the cloth constraint at most 40 shirts can be produced (M1=40), so the additional constraint for shirts is not an additional limit on x1 (If M1 were not chosen large (say M1=10), then the additional constraint for shirts would unnecessarily restrict the value of x1).

From the cloth constraint at most 53 shorts can be produced (M2=53) From the labor constraint at most 25 pants can be produced (M3=25) We thus get the mixed (binary) integer problem:

Max $6 x1 + 4 x2 + 7 x3 – 200 y1 – 150 y2 – 100 y3$

s.t. $3 x1 + 2 x2 + 6 x3 \leq 150$(Labor constraint)

$4 x1 + 3 x2 + 4 x3 \leq 160$(Cloth constraint)

$x1 \leq 40\ y1$(Shirt production constraint)

$x2 \leq 53\ y2$(Short production constraint)

$x3 \leq 25\ y3$(Pant production constraint)

$x1, x2, x3 \geq 0$ and integer

$y1, y2, y3 = 0$ or 1

Example 3.b. Lockbox

(Winston 9.2, p. 483)

Consider a national firm that receives checks from all over the United States.

There is a variable delay from when the check is postmarked (and hence the customer has met her obligation) and when the check clears (the firm can use the money).

It is in the firm's interest to have the check clear as quickly as possible since then the firm can use the money.

To speed up this clearing, firms open offices (lockboxes) in different cities to handle the checks.

Suppose firm receives payments from four regions (West, Midwest, East, and South). The average daily value from each region is as follows: $70,000 from the West,

4 www.jntufastupdates.com

$50,000 from the Midwest, $60,000 from the East, and $40,000 from the South. Firm is considering opening lockboxes in L.A., Chicago, New York, and/or Atlanta. Operating a lockbox costs $50,000 per year.

Assume that each region must send all its money to a single city. Also assume that investment rate is 20%.

Which lockboxes should firm open? (Formulate an IP that firm can use to minimize the sum of costs due to lost interest and lockbox operations.)

Answer

First we must calculate the losses due to lost interest for each possible assignment. For instance, if the West sends to New York, then on average there will be $560,000 (=8×$70.000) in process on any given day. Assuming an investment rate of 20%, this corresponds to a yearly loss of $112,000.

We can calculate the losses for the other possibilities in a similar fashion to get the following table:

Let yj be a 0-1 variable that is 1 if lockbox j is opened and 0 if it is not. Let xij be 1 if region i sends to lockbox j; and 0 otherwise.

Our objective is to minimize our total yearly costs. This is: 28 x11 + 84 x12 + … + 50 y1 + 50 y2 + 50 y3 + 50 y4

One set of constraint is that each region must be assigned to one lockbox:

$\sum_j x_{ij} = 1$ for all i

($\sum_j$ should be read as "sum over all integer values of j from 1 to n inclusive")

A region can only be assigned to an open lockbox:

$x_{1j} + x_{2j} + x_{3j} + x_{4j} \leq M\, y_j$

M is any number that should be at least 4 as there are four regions.

(Suppose we do not open LA lockbox; then y1 is 0, so all of x11, x21, x31, and x41 must also be 0. If y1 is 1, then there is no restriction on the x values.)

Min 28 x11 + 84 x12 + 112 x13 + 112 x14+ 60 x21 + 20 x22 + 50 x23 + 50 x24+ 96 x31 + 60 x32 + 24 x33 + 60 x34+ 64 x41 + 40 x42 + 40 x43 + 16 x44+ 50 y1 + 50 y2 + 50 y3 + 50 y4

s.t. x11 + x12 + x13 + x14 = 1 x21 + x22 + x23 + x24 = 1 x31 + x32 + x33 + x34 = 1 x41 + x42 + x43 + x44 = 1

$x11 + x21 + x31 + x41 \leq 4y1$ $x12 + x22 + x32 + x42 \leq 4y2$ $x13 + x23 + x33 + x43 \leq 4y3$ $x14 + x24 + x34 + x44 \leq 4y4$

All xij and yj = 0 or 1

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Membership in Specified Subsets

Using decision variables that equal 1 if an object is part of a solution and 0 otherwise, set covering, set packing, and set partitioning models formulate problems where the core issue is membership in specifed subsets.

Many applications in areas such as location problems (fire/police station, warehouse, facility), scheduling (crew, airline, truck, bus), political districting

•Set covering constraints

Require that at least one member of subcollection J belongs to a solution:

$\sum j \square J\ xj \geq 1$

•Set packing constraints

Require that at most one member of subcollection J belongs to a solution:

$\sum j \square J\ xj \leq 1$

•Set partitioning constraints

Require that exactly one member of subcollection J belongs to a solution:

$\sum j \square J\ xj = 1$

Set Covering Problems

Each member of a given set (call it set 1) must be "covered" by an acceptable member of some set (call it set 2).

The objective of a set-covering problem is to minimize the number of elements in set 2 that are required to cover all the elements in set 1.

Example 4. Fire Station

A county is reviewing the location of its fire stations. The county is made up of a number of cities:

A fire station can be placed in any city.

It is able to handle the fires for both its city and any adjacent city (any city with a non- zero border with its home city).

How many fire stations should be built and where?

Answer

We can create one variable xj for each city j (1 if we place a station in the city, 0 otherwise):

Each constraint should state that there must be a station either in city j or in some adjacent city.

The jth column of the constraint matrix represents the set of cities that can be served by a fire station in city j.

We are asked to find a set of such subsets j that covers the set of all cities in the sense that every city appears in the service subset associated with at least one fire station

Min x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

s.t. $x1 + x2 + x3 + x4 \geq 1$ (city 1)

$x1 + x2 + x3 + x5 \geq 1$ (city 2)

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x1 + x2 + x3 + x4 + x5 + x6≥ 1 (city 3)

x1+ x3 + x4+ x6 + x7≥ 1 (city 4)

x2 + x3+ x5 + x6+ x8 + x9≥ 1 (city 5)

x3 + x4 + x5 + x6 + x7 + x8≥ 1 (city 6)

x4+ x6 + x7 + x8≥ 1 (city 7)

x5 + x6 + x7 + x8 + x9 + x10≥ 1 (city 8)

x5+ x8 + x9 + x10 + x11≥ 1 (city 9)

x8 + x9 + x10 + x11x9 + x10 + x11≥ 1 (city 10)

≥ 1 (city 11)

All xj = 0 or 1

Either-Or Constraint

Given two constraints

$f(x1, x2,\ldots, xn) \leq 0(1)$

$g(x1, x2,\ldots, xn) \leq 0(2)$

ensure that at least one is satisfied (1 or 2) by adding either-or-constraints:

$f(x1, x2,\ldots, xn) \leq M\ y$

$g(x1, x2,\ldots, xn) \leq M\ (1 - y)$

Here y is a 0-1 variable, and M is a number chosen large enough to ensure that both constraints are satisfied for all values of decision variables that satisfy the other constraints in the problem:If y = 0, then (1) and possibly (2) must be satisfied.

If y = 1, then (2) and possibly (1) must be satisfied.

Example 5. Fire Station

Suppose 1.5 tons of steel and 30 hours of labor are required for production of one compact car.

At present, 6,000 tons of steel and 60,000 hours of labor are available.

For an economically feasible production, at least 1,000 cars of compact car must be produced.

•Constraint:$x \leq 0$ or $x \geq 1000$

Sign restriction: $x \geq 0$ and Integer

Answer

For f(x) = x; g(x) = 1000 – x

We can replace the constraint by the following pair of linear constraints:

$x \leq M\ y$

$1000 - x \leq M\ (1 - y)$

y = 0 or 1

M = min (6.000/1.5, 60.000/30) = 2000

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If-Then Constraints

Suppose we want to ensure that

a constraint $f(x1, x2,\ldots, xn) > 0$ implies the constraint $g(x1, x2,\ldots, xn) \geq 0$

Then we include the following constraints in the formulation:

$-g(x1, x2,\ldots, xn) \leq M\, y$ (1)

$f(x1, x2,\ldots, xn) \leq M\,(1 - y)$ (2)

Here y is a 0-1 variable, and M is a large positive number, chosen large enough so that $f<M$ and $-g<M$ hold for all values of decision variables that satisfy the other constraints in the problem.

If $f > 0$, then (2) can be satisfied only if $y = 0$. (1) implies $-g \leq 0$ or $g \geq 0$, which is the desired result

Example 6. Modified Lockbox

(Winston 9.2, p. 490)

Suppose we add the following constraint

If customers in region 1 send their payments to city 1, no other customers may send their payments to city 1:

If $x11 = 1$, then $x21 = x31 = x41 = 0$ □

If $x11 > 0$, then $x21 + x31 + x41 \leq 0$

Answer

For $f = x11$ and $g = -x21 - x31 - x41$

We can replace the implication by the following pair of linear constraints:

$x21 + x31 + x41 \leq My$ $x11 \leq M\,(1 - y)$

$y = 0$ or 1

$-g$ and f can never exceed 3, we can choose M as 3.

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