UNIT 4

DIFFERENTIAL KINEMATICS AND ROBOT DYNAMICS

SIA GROUP

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. What is "Dynamic modeling"?

Model Paper-III, Q1(e)

Answer :

In robotic concept, a manipulator within the work cycle must accelerate, move at constant speed, and decelerate according to requirements. The dynamic behavior of manipulator is expressed in terms of time-varying position and orientation. Dynamic model includes external forces exerted by environment, self weights, inertia forces, frictional forces, loads, Coriolis force due to rotation, sliding in one link and internal forces or torques. A set of mathematical equations are used to describe the dynamic behavior of the manipulator with respect to input forces or torques. The dynamic model of a manipulator is useful in computation of torque and forces required for performing a job at work station in a logical order. Dynamic model also provides information for analysis of motion of links and design of links, joints, drives and actuator. Lagrangian-Euler (LE) and Newton-Euler (NE) approaches are applied to develop the manipulator equations of motion. D'Alembert principle is also used to provide an "equivalent force dynamic model".

Q2. What is meant by "robot arm dynamics"?

Answer :

Robot Arm Dynamics

The robot arm motion, that can be determined by forming mathematical equations is called 'robot arm dynamics'. The dynamic equations describes the motion of the manipulator. These equations also helps in the robot arm motion analysis for its design and structure.

The actual dynamic model is designed based on the Newtonian laws and Lagrangian mechanics, which is used to estimate the geometry of manipulator and joint variables. The actual robot arm dynamic equations can be framed from Lagrangian-Euler and the Newton-Euler formulae. These equations also enable computer simulation of dynamics of robot arm.

Q3. Write about forward and inverse dynamics.

Answer :

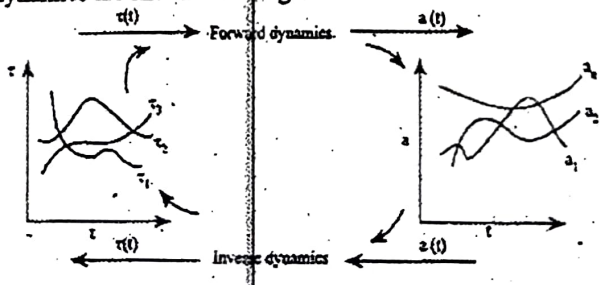
Forward Dynamics

Forward dynamics is used to determine the generalized acceleration, by analysing the forces acting on each link of the robot.

Inverse Dynamics

Inverse dynamics is used to determine the joint torques of the links and manipulator trajectory planning algorithm for the required motion of the joints.

The forward and inverse dynamics are shown in the figure.



Figure

Where,

Joint torque (τ) and acceleration (a) are the functions of time (t).

Q4. Write short notes on forward iteration and backward iteration.

Answer :

Forward Iteration

It is the recursion process, used to calculate the velocities and accelerations of all the links starting from the base to the end-effector. It is also called outward iteration. Boundary conditions of this process are linear and angular velocities are zero and the boundary acceleration is equal to the acceleration due to gravity.

Backward Iteration

It is also a recursive process, used to calculate forces and moments at each link with the help of Newton-Euler formulation. It is also called inward iteration. It is reverse of forward process, since, it starts the iteration from end-effector to the base, by considering necessary boundary conditions.

In this, the forces and moments at mass centre of each link are calculated, followed by the computation of forces and moments at each joint in reverse process.

Q5. Differentiate between Newton-Euler and Lagrangian formulations in findind the dynamic equations of motion.

Answer :

Model Paper-IV, Q1(b)

	Newton-Euler Formulation		Euler-Lagrangian Formulation
1.	It is a force balance approach.	1.	It is an energy based approach.
2.	The equation of motion is derived by computing linear and angular accelerations.	2.	The equation of motion is derived using a scalar function (Lagrangian).
3.	The computations are less complex.	3.	The computations are more complex.
4.	It is not suitable for formulation of simple physical interpretations.	4.	It is suitable for formulation of simple physical interpretations.
5.	Less time consuming method.	5.	More time consuming method.
6.	It is difficult to derive advanced control laws.	6.	It is difficult to compute real time control problems.

Q6. Define Lagrange function and state the Lagrangian expression to obtain dynamic model.

Answer :

Lagrange Function

"The algebraic difference between total kinetic energy and total potential energy of a system is defined as Lagrange function or Lagrangian". It is denoted by 'L'

i.e., $L = K.E - P.E$

The dynamic model is given by a differential equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i \text{ for } i = 1, 2, ..., n$$

Where,

L – Lagrangian

q_i and $\dot{q}_i$ – Generalized displacements of kinematic and potential energies

τ_i – Joint torque for joint-'i'

If $\tau_j = 0$, then joint 'i' is fixed.

If $\tau_i \neq 0$, then moment of manipulator is obtained by the actuator at joint-'i'

Q7. What are the singular configurations of a spherical robot?

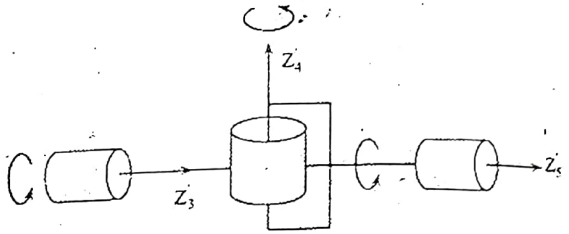
Answer :

Model Paper-I, Q1(d)

The configurations for which the rank of Jacobian (J) decreases are called as singular configurations or singularities. Since, the manipulator configurations are geometric quantities and are independent of frames, the singularities are decoupled into two forms. Thus, the singular configurations of a spherical robot are,

1. Wrist singularities
2. Arm singularities.

1. **Wrist Singularities**



Figure

Wrist singularities are caused due to the motion of spherical wrist. When the three vectors Z_3, Z_4 and Z_5 are in line relation with each other or when axes of any two revolute joints are collinear, it results in a singular configuration.

2. **Arm Singularities**

The singularities caused due to the motion of arm, consisting three or more links in the configuration is known as singularities. It can be studied by computing the Jacobian function of singular configuration of an arm.

Q8. What are the steps to be followed to obtain a dynamic equation?

Answer :

The steps to be followed to obtain the dynamic equations for a manipulator are,

1. Determination of velocity at a point in a link.
2. Determination of kinetic energy of a link.
3. Determination of potential energy of a link.
4. Obtaining the Lagrangian formula.
5. Differentiating the Lagrangian equation, gives the dynamic equations of the manipulator.

PART-B
ESSAY QUESTIONS WITH SOLUTIONS

4.1 DIFFERENTIAL TRANSFORMATION AND MANIPULATORS, JACOBIANS - PROBLEMS

Q9. What are the singularities of a manipulator? How are they classified and determined? Explain briefly.

Answer :

[April/May-13, Set-4, Q7 | Model Paper-IV, Q5(a)]

Singularities of a Manipulator

The configuration at which manipulator Jacobian 'J' becomes non-invertible is termed as Jacobian singularities. All the manipulators have singularities at boundary of their workspace and have loci of singularities inside their workspace.

Classification

Based on the location of end effector in workspace, the singular configuration is classified into two categories.

- Workspace – Boundary singularities
- Workspace – Interior singularities.

(i) Boundary Singularities

This type of singularities occurs, when the end effector is near or on the boundary of workspace. It also occurs when manipulator is fully retracted or fully stretched.

(ii) Interior Singularities

This type of singularities occurs, when the end effector is away from the boundary of workspace. These singularities are generally caused when two or more joint axes becomes collinear.

Determination of Singularities

The internal singularities can be determined by analysing the rank of Jacobian matrix. At joint variables 'q', the Jacobian matrix loses its rank and its determinant will be eliminated.

i.e., $|J| = 0$

The singularity is determined by simplifying the problem into two separate problems,

(i) Determination of singularities from motion of three joints known as arm singularities.

(ii) Determination of singularities from motion of wrist joints known as wrist singularities.

The above methods are achieved by dividing the Jacobian matrix into four sub matrices.

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Q10. What is a geometric Jacobian? Explain.

Answer :

The Jacobian is an important tool for manipulator characterization. It is useful for finding singularities, determining inverse kinematics algorithm, analysing redundancy, describing the mapping between the forces applied on end effector and torques at the respective joints.

The jacobian of any robot manipulator structure is related with end effector linear and angular cartesian velocities with the individual joint velocities.

The equation for jacobian matrix is given as,

$$V = \begin{bmatrix} V \\ W \end{bmatrix} = J(\theta).\dot{\theta}$$

Where,

$J(\theta)$ – Jacobian matrix

$\dot{\theta}$ – Joint velocity vector.

i.e., $\dot{\theta} = [\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n]^T$

'V' is the end effector linear velocity vector

i.e., $V = [V_1, V_2, V_3]^T$

'W' is the end effector angular velocity vector

i.e., $W = [W_1, W_2, W_3]^T$

The matrix which gives the relationship between the joint velocities and the corresponding angular and linear velocity of end effector is known as geometric jacobian.

Example

Consider an anthropomorphic manipulator with spherical wrist and the corresponding jacobian is of 6×6 matrix.

The jacobian matrix is determined by the following procedure,

1. The differentiation of direct kinematic function with respect to joint variables leads to the formation of analytical jacobian.

i.e., $\dot{X} = \begin{bmatrix} \dot{P} \\ \dot{\phi} \end{bmatrix} = J_A(\theta).\dot{\theta}$.

Where,

$\dot{\phi}$ – Rotational velocity

$\dot{P}$ – Translational velocity.

2. Determining the each joint velocity with the corresponding end-effector linear and angular velocities.

Thus, the above procedure leads to the formation of geometric jacobian.

Q11. What do you mean by Jacobian? Derive the Jacobian matrix for a planar 2-link revolute jointed manipulator.

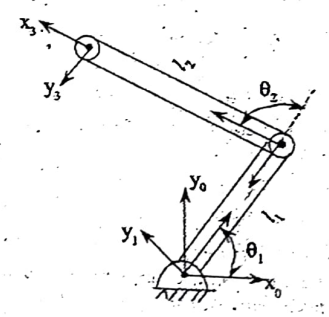
Answer :

Jacobian Matrix

Jacobian is the matrix of first-order partial derivative. For robots, the Jacobian relates the end effector velocity with the joint velocity.

Jacobian Matrix for a Planar 2-link Revolute Jointed Manipulator

Consider a two link planar manipulator, as shown in the following figure,



Figure

For two link manipulator, the link transformations are as follows,

$$^0T_0 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_1 = \begin{bmatrix} C_2 & -S_2 & 0 & l_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^3T_2 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where, l_1 and l_2 are the lengths of links (1) and (2) respectively.

The angular and linear velocities of 2-link manipulator in matrix form are given by,

$$^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$^2v_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} l_1 S_2 \dot{\theta}_1 \\ l_1 C_2 \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$^3\omega_3 = {}^2\omega_2$$

$$^3v_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} l_1 S_2 \dot{\theta}_1 \\ l_1 C_2 \dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$^3v_3 = \begin{bmatrix} l_1 S_2 \dot{\theta}_1 \\ l_1 C_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

To obtain the velocities with respect to base, the rotation matrix ${}^0R_2$ is given by.

$$
\begin{aligned}
{}^0v_3 &= {}^0R_0 \cdot {}^3v_3 \\
&= {}^0R_1 \cdot {}^1R_2 \cdot {}^2R_3 \cdot {}^3v_3 \\
{}^0v_3 &= \begin{bmatrix} -l_1 S_1\dot{\theta}_1 - l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 C_1\dot{\theta}_1 - l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}
\end{aligned}
$$

Jacobian

Consider m kinematic (end-effector) equations. Each is a function of n degrees of freedom.

$$
\begin{aligned}
x_1 &= x_1(\theta_1, \theta_2 \ldots \theta_n) \\
x_2 &= x_2(\theta_1, \theta_2 \ldots \theta_n) \\
&\vdots \quad \vdots \\
x_m &= x_m(\theta_1, \theta_2 \ldots \theta_n)
\end{aligned}
$$

In case of non-redundant manipulator $m = n$

The time derivative of the above equations is as follows,

$$
\frac{d_{x_1}}{dt} = \frac{\partial x_1}{\partial \theta_1}\cdot\frac{\partial \theta_1}{dt} + \ldots + \frac{\partial x_1}{\partial \theta_n}\cdot\frac{\partial \theta_n}{dt}
$$

$$
\vdots
$$

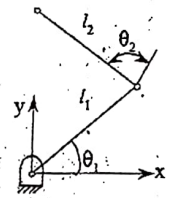
$$
\frac{d(x_m)}{dt} = \frac{\partial x_m}{\partial \theta_1}\cdot\frac{\partial \theta_1}{dt} + \ldots + \frac{\partial x_m}{\partial \theta_n}\cdot\frac{\partial \theta_n}{dt}
$$

The above equation can be written in vector form,

$$
\therefore \quad v = J\frac{d\theta}{dt}
$$

The matrix 'J' is called the "Jacobian", of which elements are the partial derivatives of kinematic equations. The relationship between the end-effector velocity and the joint velocities is fully described by the Jacobian. The end effector velocity is a linear function of the joint velocities.

$$
J = \begin{bmatrix} \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_n} \\ \vdots & & \vdots \\ \frac{\partial x_m}{\partial \theta_1} & \cdots & \frac{\partial x_m}{\partial \theta_n} \end{bmatrix}
$$



Figure

From the graphical method,

$$
\begin{aligned}
x &= l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2) \\
y &= l_1\sin\theta_1 + l_2\sin(\theta_1 + \theta_2)
\end{aligned}
$$

The Jacobian is,

$$
J = \begin{bmatrix} -l_1\sin\theta_1 & -l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 & +l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}
$$

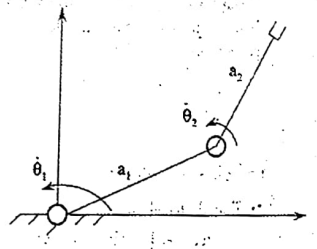
The end-effector velocity is,

$$
\frac{dx}{dt} = [-l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2)]\frac{d\theta_1}{dt} - l_2\sin(\theta_1+\theta_2)\frac{d\theta_2}{dt}
$$

$$
\frac{dy}{dt} = [l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2)]\frac{d\theta_1}{dt} + l_2\cos(\theta_1+\theta_2)\frac{d\theta_2}{dt}
$$

$\therefore$ The end effector's angular velocity is the sum of the joint velocities.

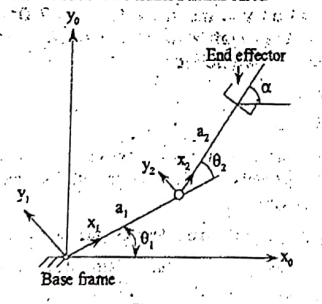
Q12. Find the Jacobian for the two link planar arm shown in figure.



Figure

Answer : Model Paper-I, Q4(b)

Jacobian Matrix for Two Link Planar Arm



Figure

Computing the Jacobian matrix that describes the relationship,

$$
\dot{x} = J_\theta\dot{\theta} \qquad \ldots (1)
$$

The end effector position and orientation is defined in the base frame by,

$$
\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

The forward kinematics gives relationship of the end effector with the joint angles as,

$$
O_{P_{2(o,x)}} = x = a_1C_1 + a_2C_{12}
$$

Where,

$$
\begin{aligned}
C_1 &= \cos\theta_1 \text{ and } C_{12} = \cos(\theta_1 + \theta_2) \\
S_1 &= \sin\theta_1 \text{ and } S_{12} = \sin(\theta_1 + \theta_2) \\
O_{P_{2(o,x)}} &= y = a_1S_1 + a_2S_{12} \\
O_{P_{2(o,\alpha)}} &= \alpha = \theta_1 + \theta_2
\end{aligned}
$$

Differentiating the above three expressions,

$$
\begin{aligned}
\dot{x} &= -a_1S_1\dot{\theta}_1 - a_2S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\
\dot{x} &= (-a_1S_1 - a_2S_{12})\dot{\theta}_1 - (a_2S_{12})\dot{\theta}_2 \\
\dot{y} &= a_1C_1\dot{\theta}_1 + a_2C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\
\dot{y} &= (a_1C_1 + a_2C_{12})\dot{\theta}_1 + (a_2C_{12})\dot{\theta}_2 \\
\dot{\alpha} &= \dot{\theta}_1 + \dot{\theta}_2
\end{aligned}
$$

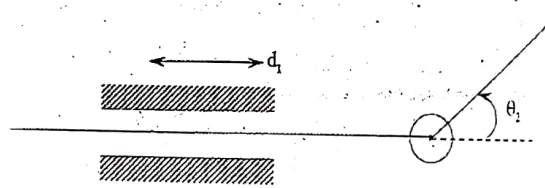
Substituting, these values in equation (1).

i.e., $\dot{x} = J_{(\theta)}\dot{\theta}$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -a_1S_1 - a_2S_{12} & -a_2S_{12} \\ a_1C_1 + a_2C_{12} & a_2C_{12} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$

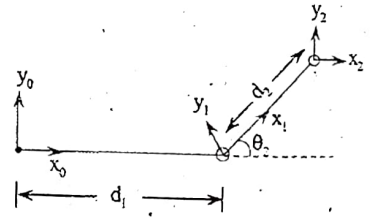
$$
\therefore \quad J = \begin{bmatrix} -a_1\sin\theta_1 - a_2\sin(\theta_1+\theta_2) & -a_2\sin(\theta_1+\theta_2) \\ a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) \\ 1 & 1 \end{bmatrix}
$$

Q13. Determine the Jacobian of the 2 DOF arm shown in figure and determine the singularities of the arm.



Figure

Answer :



Figure

Computing the Jacobian matrix that describes the relationship

$$
\dot{x} = J_\theta\cdot\dot{\theta}
$$

Let, the end effector position and orientation is defined in the base frame as,

$$
p = \begin{bmatrix} x \\ y \end{bmatrix}
$$

The forward kinematics gives relationship of the end effector to the joint angles as,

$$
\begin{aligned}
O_{P_{2(o,x)}} &= x = d_1 + d_2c_2 \\
O_{P_{2(o,y)}} &= y = d_1 + d_2s_2
\end{aligned}
$$

Differentiating the above three expression,

$$\dot{x} = -d_2 s_2 \dot{\theta}_1$$

$$\dot{y} = d_2 c_2 \dot{\theta}_2$$

Substitute, $\dot{x} = J_\theta . \dot{\theta}$ in the matrix form,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}_{2\times1} = \begin{bmatrix} 0 & -d_2 s_2 \\ 0 & d_2 c_2 \end{bmatrix}_{2\times2} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}_{2\times1}$$

The Jacobian provides a linear transformation that gives both velocity and a force map for a robot manipulator. The equation is insignificant for above case.

For finding singularities of the arm,

$$\dot{\theta} = J(\theta)^{-1} \dot{x}$$

But, $J(\theta) = \begin{bmatrix} 0 & -d_2 s_2 \\ 0 & d_2 c_2 \end{bmatrix}$

$$J(\theta)^{-1} = \frac{1}{\det|J(\theta)|} Adj(J(\theta))$$

Consider the determinant matrix of order 2 × 2

$$\det|J(\theta)| = \begin{vmatrix} 0 & -d_2 s_2 \\ 0 & d_2 c_2 \end{vmatrix}$$

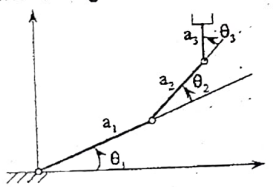
$$= 0(d_2 c_2) - (-d_2 s_2) = 0$$

$$\therefore \det|J(\theta)| = 0$$

Therefore, $\det|J(\theta)|$ is not a function of θ_1.

The singularities for $\det|J|$ at joint 2 i.e., θ_2 varies from 0 to π.

Q14. Compute the Jacobian for the three link planar arm shown in figure.

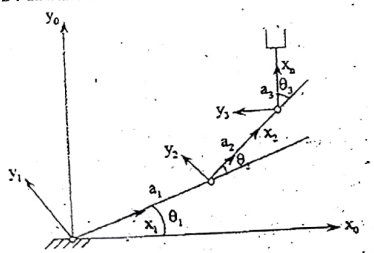


Figure

Nov.-10, Set-4, Q5

Answer :

Jacobian Matrix for Three Link Planar Arm



Figure

Computing the Jacobian matrix that describes the relationship as,

$$\dot{x} = J_\theta \dot{\theta} \quad \ldots (1)$$

The end effector position and orientation is defined in the base frame by,

$$\bar{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The forward kinematics gives relationship of the end effector to the joint angles as,

$$O_{P_{3(o,x)}} = x = a_1 C_1 + a_2 C_{12} + a_3 C_{123}$$

Where,

$$C_1 = \cos\theta_1$$

$$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$$

$$S_1 = \sin\theta_1$$

$$S_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$$

$$O_{P_{3(o,y)}} = y = a_1 S_1 + a_2 S_{12} + a_3 S_{123}$$

$$O_{P_{3(o,\alpha)}} = \alpha = \theta_1 + \theta_2 + \theta_3$$

Differentiating the above three expressions,

$$\dot{x} = -a_1 S_1 \dot{\theta}_1 - a_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2) - a_3 S_{123}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$\dot{x} = (-a_1 S_1 - a_2 S_{12} - a_3 S_{123})\dot{\theta}_1 - (a_2 S_{12} + a_3 S_{123})\dot{\theta}_2 - (a_3 S_{123})\dot{\theta}_3$$

$$\dot{y} = a_1 C_1 \dot{\theta}_1 + a_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2) + a_3 C_{123}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$\dot{y} = (a_1 C_1 + a_2 C_{12} + a_3 C_{123})\dot{\theta}_1 + (a_2 C_{12} + a_3 C_{123})\dot{\theta}_2 + a_3 C_{123}\dot{\theta}_3$$

$$\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

Substituting the values i.e., ($\dot{x}$, $\dot{y}$ and $\dot{\alpha}$) in equation (1),

i.e., $\dot{x} = J_{(\theta)}\dot{\theta}$

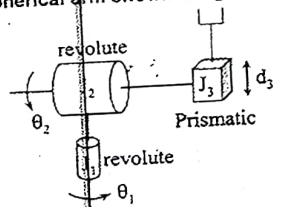
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = 2\begin{bmatrix} -a_1 S_1 - a_2 S_{12} - a_3 S_{123} & -a_2 S_{12} - a_3 S_{123} & -a_3 S_{123} \\ a_1 C_1 + a_2 C_{12} + a_3 C_{123} & a_2 C_{12} + a_3 C_{123} & a_3 C_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Therefore, Jacobian for the three link planar arm is given by,

$$\therefore J = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) - a_3 \sin(\theta_1 + \theta_2 + \theta_3) & \cdot \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3) & \\ 1 & \end{bmatrix}$$

$$\begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) - a_3 \sin(\theta_1 + \theta_2 + \theta_3) & -a_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3) & a_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ 1 & 1 \end{bmatrix}$$

Q15. Compute the Jacobian for the spherical arm shown in figure.



Figure

[Nov.-11, Set-3, Q5 | Model Paper-II, Q2(b)]

Answer :

D-H parameter for the given spherical arm.

Link	α	d	θ
1	$\frac{-\pi}{2}$	0	θ_1
2	$\frac{\pi}{2}$	0	θ_2
3	0	d_3	0

The overall homogeneous transformation matrix is given by,

$${}^0T_3 = {}^0T_1 . {}^1T_2 . {}^2T_3$$

$${}^0T_3 = \underbrace{\begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^0T_1} \underbrace{\begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^1T_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^2T_3}$$

$$= \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & 0 \\ S_1C_2 & C_1 & S_1S_2 & 0 \\ -S_2 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & C_1S_2d_3 \\ S_1C_2 & C_1 & S_1S_2 & S_1S_2d_3 \\ -S_2 & 0 & C_2 & C_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Considering the last column of homogeneous matrix,

$$x = C_1S_2d_3$$

$$y = S_1S_2d_3$$

$$z = C_2d_3$$

On differentiating the above equation,

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial\theta_1}\dot{\theta}_1 + \frac{\partial x}{\partial\theta_2}\dot{\theta}_2 + \frac{\partial x}{\partial d_3}\dot{d}_3$$

$$= [-S_1S_2d_3]\dot{\theta}_1 + [C_1C_2d_3]\dot{\theta}_2 + [C_1S_2]\dot{d}_3$$

$$\dot{y} = \frac{dy}{dt} = \frac{\partial y}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial y}{\partial \theta_2}\dot{\theta}_2 + \frac{\partial y}{\partial d_3}\dot{d}_3$$

$$= [-C_1 S_2 d_3]\dot{\theta}_1 + [S_1 C_2 d_3]\dot{\theta}_2 + [S_1 S_2]\dot{d}_3$$

$$\dot{z} = \frac{dz}{dt} = \frac{\partial z}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial z}{\partial \theta_2}\dot{\theta}_2 + \frac{\partial z}{\partial d_3}\dot{d}_3$$

$$= 0 + [-S_2 d_3]\dot{\theta}_2 + [C_2]\dot{d}_3$$

Rearranging the above equations in matrix form,

$$\therefore \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}-S_1 S_2 d_3 & C_1 C_2 d_3 & C_1 S_2\\ C_1 S_2 d_3 & S_1 C_2 d_3 & S_1 S_2\\ 0 & -S_2 d_3 & C_2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{d}_3\end{bmatrix}$$

Therefore, Jacobian matrix for the given spherical arm is given by,

$$J = \begin{bmatrix}-S_1 S_2 d_3 & C_1 C_2 d_3 & C_1 S_2\\ C_1 S_2 d_3 & S_1 C_2 d_3 & S_1 S_2\\ 0 & -S_2 d_3 & C_2\end{bmatrix}$$

Q16. Determine the Jacobian of the 3 DOF Euler wrist and also determine the singularities of the wrist.

Answer :

The singular configurations of planar mechanism can be described by degeneracies of the corresponding velocity equation which is of the form,

$$J_q \dot{q} + J_\theta \dot{\theta} = 0$$

Where,

$q - [x, y, \phi]^T$ is the vector of generalized coordinates defining the pose of the planar (mobile) platform.

θ – Vector of input coordinates

J_q and J_θ are 3 × 3 Jacobian matrices.

Two main types of singularities are defined.

Type I: when $\det |J_\theta| = 0$

Type II when $\det |J_q| = 0$

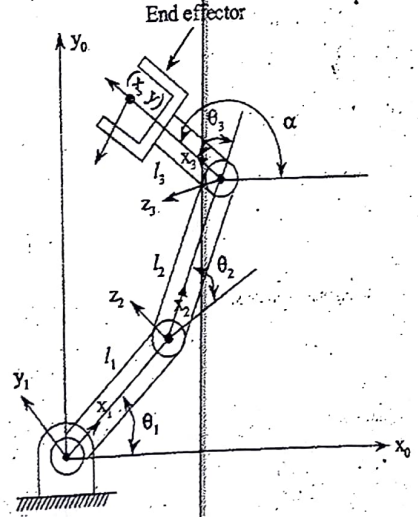


Figure: 3-DOF Planar Manipulator with Revolute Joints

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Computing the Jacobian matrix that describes the relationship as.

$$\dot{x} = J_\theta \dot{\theta}$$

The end effector position and orientation is defined in the base frame by,

$$\bar{x} = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

The forward kinematics gives relationship of the end effector to the joint angles as.

$$O_{p_{3(x,y)}} = x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$O_{p_{3(x,y)}} = y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$O_{p_{3(x,y)}} = \alpha = \theta_1 + \theta_2 + \theta_3$$

Differentiating the above three expressions gives,

$$\dot{x} = -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) - l_3 s_{123}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$= -(l_1 s_1 + l_2 s_{12} + l_3 s_{123})\dot{\theta}_1 - (l_2 s_{12} + l_3 s_{123})\dot{\theta}_2 - (l_3 s_{123})\dot{\theta}_3$$

$$\dot{y} = l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) + l_3 c_{123}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$= (l_1 c_1 + l_2 c_{12} + l_3 c_{123})\dot{\theta}_1 + (l_2 c_{12} + l_3 c_{123})\dot{\theta}_2 + (l_3 c_{123})\dot{\theta}_3$$

$$\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

$$\dot{x} = J_{(\theta)}\dot{\theta}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}-l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123}\\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123}\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix}$$

The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the above case, the equations are insignificant.

The end effector moves with a certain velocity in Cartesian space. Using linear transformation, relating the joint velocity to the Cartesian velocity, joint rates at each instance along the path.

$$\dot{\theta} = J(\theta)^{-1}\dot{x}$$

$$\det |J(\theta)| = \begin{bmatrix}-l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123}\\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123}\\ 1 & 1 & 1\end{bmatrix}$$

$$= l_1 l_2 s_2$$

$$\det |J(\theta)| = l_1 l_2 s_2 = 0$$

$\therefore$ $\det[J(\theta)]$ is not a function of θ_1, θ_3.

Q17. For a planar RP manipulator, derive the Jacobian matrix and find the singularities.

Answer :

Consider a planar manipulator in motion. By performing velocity analysis of a mechanism, it is convenient to define a matrix quantity called the "Jacobian" of the manipulator. The Jacobian specifies a mapping from velocities in joint space to velocities in Cartesian space.

(i) The geometrical relationship between joint rates and velocity of the end effector described in a matrix is called 'Jacobian'

(ii) The nature of Jacobian mapping changes, as the configuration of the manipulator changes. At Cartesian points, called 'singularities', this mapping is non-invertible.

The Jacobian is a multidimensional form of the derivative. The vector rotation and six functions are as follows,

$$y = F(x)$$

$$y_1 = f_1(x_1, x_2, x_3 \ldots x_6)$$
$$y_2 = f_2(x_1, x_2, x_3 \ldots x_6)$$
$$\vdots \quad \vdots \quad \vdots$$
$$y_6 = f_6(x_1, x_2, x_3 \ldots x_6)$$

Differentials of y_i with respect to x_j

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}.\delta x_1 + \ldots + \frac{\partial f_1}{\partial x_6}.\delta x_6$$
$$\vdots \quad \vdots \quad \vdots$$
$$\delta y_6 = \frac{\partial f_6}{\partial x_1}.\delta x_1 + \ldots + \frac{\partial f_6}{\partial x_6}.\delta x_6$$
$$\therefore \quad \delta y = \frac{\partial F}{\partial X}.\delta x$$

The 6×6 matrix of partial derivatives in above relation is called Jacobian matrix.

$$\delta y = J(X).\delta X$$

Differentiate both sides with respect to time,

$$\dot{y} = J(X).\dot{X}$$

For planar *RP* manipulator, Jacobian relates joint velocities to Cartesian velocities of the tip of the arm.

Example

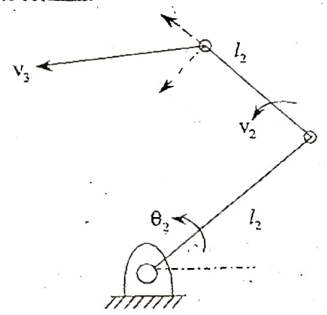
$$0_v = 0_{J(\theta)}.\dot{\theta}$$

Where,

θ = Vector of joint angles of manipulator

v = Vector of Cartesian velocities.

In Jacobian matrix, the number of rows is equal to the number of degrees of freedom in Cartesian space and number of columns is equal to the number of joints of the manipulator. For planar manipulator, Jacobian should have more than 3 rows and multiple columns.



$$1_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2_{v_2} = \begin{bmatrix} l_1 S_1 \dot{\theta}_1 \\ l_1 C_2 \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$3_{v_3} = \begin{bmatrix} l_1 S_1 \dot{\theta}_1 \\ l_1 C_2 \theta_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$0_{v_3} = \begin{bmatrix} -l_1 S_1 \dot{\theta}_1 - l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 C_2 \dot{\theta}_1 + l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$0_{J(\theta)} = \begin{bmatrix} -l_1 S_1 & -l_2 S_{12} & -l_1 S_{12} \\ l_1 C_1 & +l_2 C_{12} & l_2 C_{12} \end{bmatrix}$$

Singularities

Manipulators have values of θ where Jacobian becomes singular. Such locations are called singularities. All the manipulators have singularities at the boundary of their workspace and have loci of singularities inside their workspace. Singularities are classified as,

(i) Boundary singularities

(ii) Interior singularities.

Examine determinant of Jacobian matrix,

$$\therefore \quad \det|J(\theta)| = \begin{bmatrix} l_1 S_2 & 0 \\ l_1 C_1 + l_2 & l_2 \end{bmatrix} = l_1 l_2 S_2 = 0$$

Q18. For a planar RR manipulator, find the linear velocity and angular velocity of the end effector.

Answer :

Linear and Angular Velocity of the End Effector

Let the end effector (P) has frame of reference {2}. Velocity of the point P in frame {2} can be expressed with respect to both frame {1} and frame {0}. RR manipulator is as shown in the following figure.

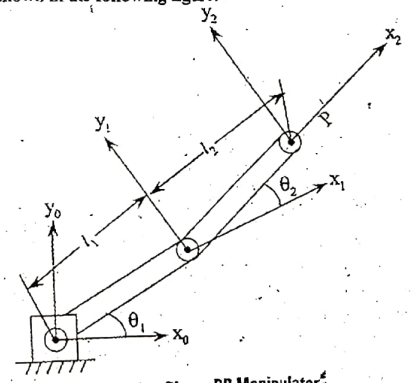


Figure: Planar RR Manipulator

Let, $l_1 = l_2 = 1$

Link transformation matrices are as follows,

$$^0T_1 = \begin{bmatrix} C_1 & -S_1 & 0 & C_1 \\ S_1 & C_1 & 0 & S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = \begin{bmatrix} C_2 & -S_2 & 0 & C_2 \\ S_2 & C_2 & 0 & S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation matrix of frame {2} relative to frame {0} is obtained as follows,

$$^0T_2 = \begin{bmatrix} C_1 C_2 - S_1 S_2 & -C_1 S_2 - S_1 C_2 & 0 & C_1 + C_1 C_2 - S_1 S_2 \\ C_1 S_2 + S_1 C_2 & C_1 C_2 - S_1 S_2 & 0 & S_1 + C_1 S_2 + S_1 C_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & C_1 + C_{12} \\ S_{12} & C_{12} & 0 & S_1 + S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$C_i = \cos\theta_i$

$S_i = \sin\theta_i$

$C_{12} = \cos(\theta_1 + \theta_2)$

$S_{12} = \sin(\theta_1 + \theta_2)$

Linear velocity of link –1 is given by,

$$V_1 = \frac{\partial(^0T_1)}{\partial\theta_1}\ \dot{\theta}_1\ ^1D_1$$

Where,

$$\frac{\partial(^0T_1)}{\partial\theta_1} = \begin{bmatrix} -S_1 & -C_1 & 0 & -S_1 \\ C_1 & -S_1 & 0 & C_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$^1D_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And, $^1D_1$ – Origin of frame {1}

$$\therefore \quad V_1 = \begin{bmatrix} -S_1 \\ -C_1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1$$

Angular velocity, (ω_1)

$$\omega_1 = \omega_0 + {}^0R_0 \hat{Z}_0 \dot{\theta}_1$$

Where, $\omega_0 = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\hat{Z}_0 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$${}^0R_0 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\therefore \quad \omega_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}$$

Velocity of link-2 relative to base frame is obtained as follows,

$$V_2 = \frac{\partial({}^0T_2)}{\partial\theta_1}\dot{\theta}_1\,{}^2D_2 + \frac{\partial({}^0T_2)}{\partial\theta_2}\dot{\theta}_2\,{}^2D_2$$

${}^2D_2$– Origin of frame 2 $= \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$$\frac{\partial({}^0T_2)}{\partial\theta_1} = \begin{bmatrix}-S_1C_2 - C_1S_2 & S_1S_2 - C_1C_2 & 0 & -S_1 - S_1C_2 - C_1S_2\\ -S_1S_2 + C_1C_2 & -S_1C_2 - C_1S_2 & 0 & C_1 - S_1S_2 + C_1S_2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\frac{\partial({}^0T_2)}{\partial\theta_2} = \begin{bmatrix}-C_1S_2 - S_1C_2 & -C_1C_2 + S_1S_2 & 0 & -C_1S_2 - S_1C_2\\ C_1C_2 - S_1S_2 & -C_1S_2 - S_1C_2 & 0 & C_1C_2 - S_1S_2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\therefore \quad V_2 = \begin{bmatrix}-S_1\dot{\theta}_1 - S_{12}(\dot{\theta}_1 + \dot{\theta}_2)\\ C_1\dot{\theta}_1 + C_{12}(\dot{\theta}_1 + \dot{\theta}_2)\\ 0\end{bmatrix}$$

Angular velocity, (ω_2)

$$\omega_2 = \omega_1 + {}^0T_1 Z_1 \dot{\theta}_2$$

$$= \begin{bmatrix}0\\0\\\dot{\theta}_1\\0\end{bmatrix} + \begin{bmatrix}C_1 & -S_1 & 0 & C_1\\ S_1 & C_1 & 0 & S_1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\dot{\theta}_2$$

$$\therefore \quad \omega_2 = \begin{bmatrix}0\\0\\(\dot{\theta}_1 + \dot{\theta}_2)\end{bmatrix}$$

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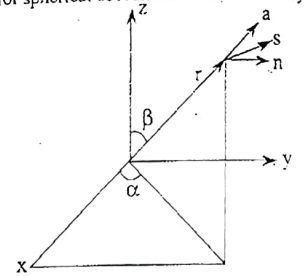
Q19. Find the manipulator Jacobian matrix J(q) of the five axis spherical coordinate robot.

Answer :

For kinematic analysis of robots, which are a set of rigid links connected together at various joints, a manipulator Jacobian matrix J(q) is used. Generally a manipulator consists of.

(i) Waist rotation
(ii) Shoulder rotation
(iii) Elbow rotation
(iv) Flange rotation and
(v) Wrist rotations.

The coordinate frame arrangement for spherical coordinate axis (five axis system) is shown in figure below.



Figure

Orientations

Roll, pitch, yaw

Positions

Translation of r units along the OZ axis, rotation of 'β' angle about OY axis and a rotation of 'α' angle about OZ axis

Establishing the joint coordinate frames using the D-H convention (Denavit-Hartenberg representation), as shown in above figure.

The link parameters are shown in the table below.

Link	d_i	a_i	α_i	θ_i
1	0	0	-90°	–
2	d_2	0	$+90^\circ$	–
3	–	0	0	0
4	0	0	-90°	–
5	0	0	$+90^\circ$	–
6	0	0	d_6	–

Table: D-H Parameters

The matrices A_i

$$A_1 = \begin{bmatrix}C_1 & 0 & -S_1 & 0\\ S_1 & 0 & C_1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$A_2 = \begin{bmatrix}C_2 & 0 & S_2 & 0\\ S_2 & 0 & -C_2 & 0\\ 0 & 1 & 0 & d_2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$A_3 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_3\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$A_4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_6^0 is given as, $A_1 \ldots\ldots A_6$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \end{bmatrix}$$

Where,

$$r_{11} = C_1[C_2(C_4 C_5 C_6 - S_4 S_6) - S_2 S_5 C_6] - S_1(S_4 C_5 C_6 + C_4 S_6)$$

$$r_{21} = S_1[C_2(C_4 C_5 C_6 - S_4 S_6) - S_2 S_5 C_6] + C_1(S_4 C_5 C_6 + C_4 S_6)$$

$$r_{31} = -S_2(C_4 C_5 C_6 - S_4 S_6) - C_2 S_5 C_6$$

$$r_{12} = C_1[-C_2(C_4 C_5 C_6 + S_4 C_6) + S_2 S_5 S_6] - S_1(-S_4 C_5 S_6 + C_4 C_6)$$

$$r_{22} = S_1[-C_2(C_4 C_5 S_6 + S_4 C_6) + C_2 S_5 S_6] + C_1(-S_4 C_5 S_6 + C_4 C_6)$$

$$r_{32} = -S_2[C_4 C_5 S_6 + S_4 C_6] + C_2 S_5 S_6$$

$$r_{13} = C_1(C_2 C_4 C_5 + S_2 C_5) - S_1 S_4 S_5$$

$$r_{23} = S_1(C_2 C_4 S_5 + S_2 C_5) + C_1 S_4 S_5$$

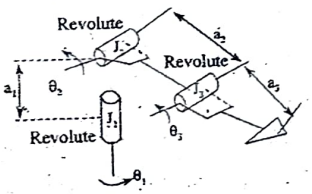
$$r_{33} = -S_2 C_4 S_5 + C_2 C_5$$

$$d_x = C_1 S_2 d_3 - S_1 d_2 + d_6(C_1 C_2 C_4 C_5 + C_1 C_5 S_2 - S_1 S_4 S_5)$$

$$d_y = S_1 S_2 d_3 + C_1 d_2 + d_6(C_1 S_4 S_5 + C_2 C_4 S_1 S_5 + C_5 S_1 S_2)$$

$$d_z = C_2 d_3 + d_6(C_2 C_5 - C_4 S_2 S_5)$$

Q20. Determine the Jacobian matrix which calculates linear velocity of the tool tip from the three joint rates for the manipulator shown in figure. Joints J_1 and J_2 are perpendicular and joints J_2 and J_3 are parallel.



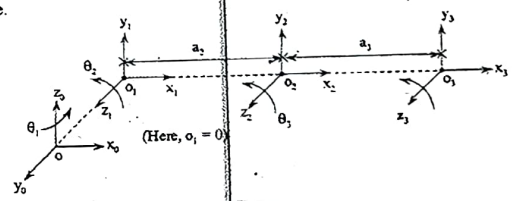
Figure

[Nov.-11, Set-2, Q5 | Model Paper-II, Q6(b)]

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Answer :

Let, a_1, a_2 and a_3 are the lengths of respective links. Therefore, to calculate the position and orientation of arm, consider the frame, as shown in figure.



Figure

From the above figure, joint link parameters for articulated arm can be formulated as,

Link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Transformation matrix can be written as,

$$^{i-1}U_i = \begin{bmatrix} C\theta_i & -S\theta_i \cdot C\alpha_i & S\theta_i S\alpha_i & \alpha_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the transformation matrix for link-1,

$$^0U_i = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for link-2,

$$^1U_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for link-3,

$$^2U_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3 C_3 \\ S_3 & C_3 & 0 & a_3 S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation matrix for the end point of articulated arm is given as,

$$^0U_3 = {}^0U_1 \times {}^1U_2 \times {}^2U_3$$

On multiplying,

$$^0U_3 = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & C_1(a_3 C_{23} + a_2 C_2) \\ S_1 C_{23} & -S_1 S_{23} & -C_1 & S_1(a_3 C_{23} + a_2 C_2) \\ S_{23} & C_{23} & 0 & a_3 S_{23} + a_2 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$C_{23} = \cos(\theta_2 + \theta_3)$

$S_{23} = \sin(\theta_2 + \theta_3)$

At initial position, i.e., $\theta_1 = \theta_2 = \theta_3 = 0$.

Jacobian Matrix

As the given manipulator consists of three revolute joints, first determine the Jacobian matrix for each joint and then find the total Jacobian matrix.

The transformation of the unit vector $\bar{u}$ with respect to the base frame $\{0\}$ is given by the vector $P_1 = {}^1R_0\bar{u}$

And, $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ [$\because$ Rotation matrix ${}^0R_1$ and transformation matrix ${}^1u_1$ matrix are identity matrices]

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Position vector of the end-effector ($i = 1$ and $n = 3$) is given by,

$${}^{n-1}P_n = {}^{n-1}U_n O_n - {}^{n-1}U_{n-1}O_{n-1}$$

$${}^2P_3 = {}^2U_3O_3 - {}^2U_2O_2$$

Substituting the value of ${}^2U_3$,

$${}^2P_3 = \begin{bmatrix} C_{23} & -C_{23}S_{23} & S_{23} & C_{23}(a_3C_{23} + a_2C_2) \\ S_{23}C_{23} & -S_{23}S_{23} & -C_{23} & S_{23}(a_3C_{23} + a_2C_2) \\ S_{23} & C_{23} & 0 & a_3S_{23} + a_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^2P_3 = \begin{bmatrix} C_{23}(a_3C_{23} + a_2C_2) \\ S_{23}(a_3C_{23} + a_2C_2) \\ a_3S_{23} + a_2S_2 \\ 0 \end{bmatrix}$$

The first column of the Jacobian J_1 with revolute joint is computed from the formula.

$$J_1 = \begin{bmatrix} P_{n-1} \times {}^{n-1}P_n \\ P_{n-1} \end{bmatrix}$$

For revolute joint (1), the Jacobian J_1 is determined as,

$$J_1 = \begin{bmatrix} P_{n-1} \times {}^{n-1}P_n \\ P_{n-1} \end{bmatrix} \quad [\because i = 1 \text{ and } n = 3]$$

$$J_1 = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} C_{23}(a_3C_{23} + a_2C_2) \\ S_{23}(a_3C_{23} + a_2C_2) \\ a_3S_{23} + a_2S_2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$\Rightarrow \quad J_1 = \begin{bmatrix} -S_1(a_3C_{23} + a_2C_2) \\ C_1(a_3C_{23} + a_2C_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly Jacobian matrix for second and third revolute joints can be obtained as,

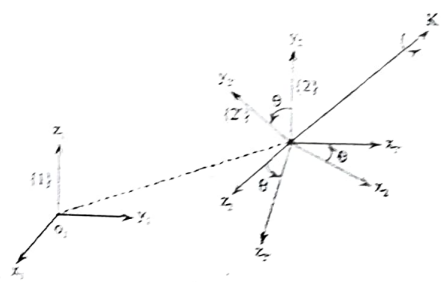
$$J_2 = \begin{bmatrix} -C_1(a_3S_{23} + a_2S_2) \\ -S_1(a_3S_{23} + a_2S_2) \\ a_3C_{23} + a_2C_2 \\ S_1 \\ -C_1 \\ 0 \end{bmatrix} \quad \text{and} \quad J_3 = \begin{bmatrix} -a_3C_1S_{23} \\ -a_3S_1S_{23} \\ a_3C_{23} \\ S_1 \\ -C_1 \\ 0 \end{bmatrix}$$

$\therefore$ The total Jacobian matrix is obtained by combining the columns of the three revolute joints.

$$\therefore \quad J = \begin{bmatrix} -S_1(a_3C_{23} + a_2C_2) & -C_1(a_3S_{23} + a_2S_2) & -a_3C_1S_{23} \\ C_1(a_3C_{23} + a_2C_2) & -S_1(a_3S_{23} + a_2S_2) & -a_3S_1S_{23} \\ 0 & a_3C_{23} - a_2C_2 & a_3C_{23} \\ 0 & S_1 & S_1 \\ 0 & -C_1 & -C_1 \\ 1 & 0 & 0 \end{bmatrix}$$

Q21. Show that the three differential rotations of δx, δy and δz made in any order about the x, y, and z axes respectively are equivalent to a differentional rotation of $d\theta$ about axis K.

Answer : (April/May-13, Set-3, Q8, Model Paper-VIII)



Figure

Consider frame {2} attached to a rigid body and it is rotated about frame {1} at time 't' and ($t + dt$) respectively. The transformation matrix with respect to time 't' is given by,

$${}^1T_2 = \lim_{dt \to 0} \frac{{}^1T_2 - {}^1T_2}{dt}$$

The frame {2'} is obtained by considering the fixed reference frame {1} and rotating frame {2} about an axis [K] during the time interval 'dt'. Thus, rotation of frame {2} with respect to frame {2'} by an angle 'θ' is given as,

$$T_K(\theta) = \begin{bmatrix} K_x^2V(\theta) + C\theta & K_xK_yV(\theta) - K_zS(\theta) & K_xK_zV(\theta) + K_yS(\theta) & 0 \\ K_xK_yV(\theta) + K_zS(\theta) & K_y^2V(\theta) + C\theta & K_yK_zV(\theta) - K_xS(\theta) & 0 \\ K_xK_yV(\theta) - K_yS(\theta) & K_yK_zV(\theta) + K_xS(\theta) & K_z^2V(\theta) + C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$C\theta = \cos\theta$, $S\theta = \sin\theta$, $V(\theta) = 1 - \cos\theta$ and $K = [K_x, K_y, K_z]^T$

If 'θ' is considered as small, then $\sin\theta \simeq d\theta$ and $\cos\theta \simeq 1$.

$\therefore (d\theta)^2 = 0$

Thus equation (2) reduces to,

$$T_K(d\theta) = \begin{bmatrix} 1 & -K_z \cdot d\theta & K_y \cdot d\theta & 0 \\ K_z \cdot d\theta & 1 & -K_x \cdot d\theta & 0 \\ -K_y \cdot d\theta & K_x \cdot d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (3)$$

The frame $\{2'\}$ is obtained by considering homogeneous transformation relative to frame 'A'

i.e., $^1T_2 = T_K(d\theta) \cdot {}^1T_2(t)$ ... (4)

Substituting equation (4) in (1),

$$^1\dot{T}_2 = \lim_{dt \to 0} \left(\frac{T_K(d\theta) - I}{dt} \right) {}^1T_2(t) \quad \ldots (5)$$

Where,

I is identity matrix of size 4×4 matrix.

Substituting $T_K(d\theta)$ from equation (3) in (5),

$$^1\dot{T}_2 = \lim_{dt \to 0} \begin{bmatrix} 0 & -K_z \frac{d\theta}{dt} & K_y \frac{d\theta}{dt} & 0 \\ K_z \frac{d\theta}{dt} & 0 & -K_x \frac{d\theta}{dt} & 0 \\ -K_y \frac{d\theta}{dt} & K_x \frac{d\theta}{dt} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^1T_2(t)$$

Let, $\dot{\theta}$ be the speed of angular rotation of frame {2} about K-axis.

i.e., $\dot{\theta} = \lim_{dt \to 0} \left(\frac{d\theta}{dt} \right)$

$$^1\dot{T}_2 = \begin{bmatrix} 0 & -K_z\dot{\theta} & K_y \cdot \dot{\theta} & 0 \\ K_z \cdot \dot{\theta} & 0 & -K_x \cdot \dot{\theta} & 0 \\ -K_y \cdot \dot{\theta} & K_x \cdot \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^1T_2(t)$$

$$^1\dot{T}_2 = \begin{bmatrix} 0 & -K_z & K_y & 0 \\ K_z & 0 & -K_x & 0 \\ -K_y & K_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta} \, {}^1T_2(t) \quad \ldots (6)$$

Let, the vector '$^1\delta_2$' is directed along K-axis.

The vector '$^1\delta_2$' is equivalent to three rotations of δx, δy and δz about x, y and z axes respectively.

$$\therefore \quad ^1\delta_2 = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} \dot{\theta}$$

∴ The three differential rotation of δx, δy and δz in any order about x, y and z axes are equivalent to differential rotation about K-axis.

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4.2 DYNAMICS - LAGRANGE, EULER AND NEWTON, EULER FORMULATIONS - PROBLEMS

Q22. Establish the dynamic model of a one-axis robot (Inverted Pendulum) with Lagrangian-Euler formulation.

Answer :

The dynamic model for the dynamic behavior of the manipulator is developed by the Lagrangian-Euler (L.E) approach, which is an energy-based approach, considering the rigid body motion.

Lagrange function or Lagrangian (L) is a scalar function and is defined as "the difference between the total $K.E$ (K) and the total $P.E$ (P) of a mechanical system".

$$\therefore \quad L = K - P$$

The Lagrangian-Euler dynamic formulation is based on a set of generalized coordinates to describe the system variables. The dynamic model based on Lagrangian-Euler formulation is obtained from the Lagrangian, as a set of equations.

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{q}_i}\right) - \frac{\partial l}{\partial q_i} = \tau_i \text{ (for } i = 1, 2 \ldots n.)$$

The left hand side of dynamic equation can be interpreted as a sum of the torques or forces due to kinetic energy and potential energy present in the system. The right hand side τ_i is the joint torque for joint i that is displaced by the actuator i. If $\tau_i = 0$, it means that joint i does not move and if $\tau_i \neq 0$, the manipulator movement is modified by the actuator at joint i.

Dynamic Model of a One-axis Robot

Figure below shows a single-link pendulum torque transmission between motor and load link.

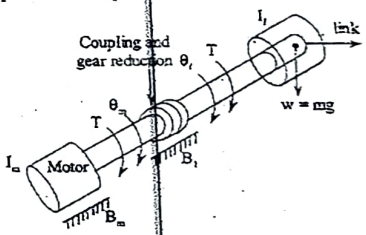


Figure (1)

Let,

$\dot{\theta}_l$ – Angular displacement of the link

$\dot{\theta}_m$ – Angular displacement of the motor shaft

Gear ratio or speed ratio of power transmission between motor and link.

$$\frac{\omega_m}{\omega_l} = \frac{\dot{\theta}_m}{\dot{\theta}_l} = \frac{n}{1} = n{:}1$$

Applying Lagrangian-Euler formulation,

Kinetic energy of the system $(K) = \frac{1}{2} I_m \omega_m^2 + \frac{1}{2} I_l \omega_l^2$

Where,

$\omega_m = \frac{d}{dt}(\dot{\theta}_m) = \theta_m$ = Angular speed of motor

$\omega_l = \frac{d}{dt}(\dot{\theta}_l) = \theta_l$ = Angular speed of link.

I_m = Mass moment of inertia of motor (kg-m^2)

I_l = Mass moment of inertia of link (kg-m^2)

$$\therefore \quad K = \frac{1}{2} I_m \dot{\theta}_m^2 + \frac{1}{2} I \left(\frac{\dot{\theta}_m}{\pi} \right)^2$$

$$K = \frac{1}{2} \dot{\theta}_m^2 \left(I_m + \frac{I}{\pi^2} \right)$$

The potential energy (P) = mgh

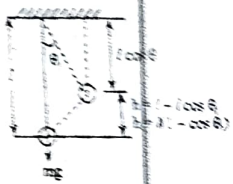


Figure (2)

$$\therefore \quad P = mgh$$
$$= mgl(1 - \cos\theta)$$
$$= mgl(1 - \cos(\theta_m/\pi))$$

Where,

m – Total mass of the link pendulum (kg)

l – Distance from the joint axis to the link center of mass i.e., change in position of link mass center (m).

The Lagrangian (L) = $K - P$

$$= \frac{1}{2}\left(I_m + \frac{I}{\pi^2} \right) \dot{\theta}_m^2 - mgl\left(1 - \cos\left(\frac{\theta_m}{\pi} \right) \right)$$

Substituting the above expression into the Euler-Lagrangian equation, yields the equation of motion of link.

$$\left(I_m + \frac{I}{\pi^2} \right) \ddot{\theta}_m + B\dot{\theta}_m + K \sin\left(\frac{\theta_m}{\pi} \right) = \tau$$

Where,

$\dot{\theta}_m$ – Viscous damping coefficient of system (Ns/m)

K – Stiffness of system (N/m)

τ – Motor input torque (N-m).

Let,

$B_m \dot{\theta}_m$ = Non-conservative damping torque

$B_l \dot{\theta}_l$ = Reflecting torque of link damping

$$T = \tau - \left(B_m + \frac{B_l}{\pi^2} \right) \dot{\theta}_m$$

The complete expression for the dynamic model of the system is,

$$\boxed{I\ddot{\theta}_m + B\dot{\theta}_m + K \sin\left(\frac{\theta_m}{\pi} \right) = \tau}$$

Where,

$$J = I_m + \frac{I_l}{\pi^2} \text{ kg-m}^2$$

$$B = B_m + \frac{B_l}{\pi^2} \text{ Ns/m}$$

$$K = \frac{B_l}{\pi^2} \text{ N/m}.$$

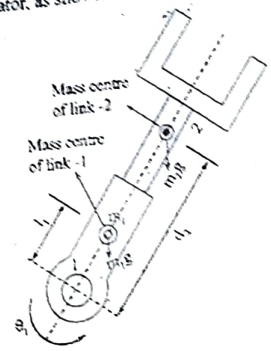
The above expression is a second order non-linear differential equation in generalized coordinate Euler-Lagrangian form.

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Q23. Using Lagrangian-Euler formulation, derive the expression for the joint torques or forces of a planar R P robotic manipulator.

Answer :

Consider a planar R P robotic manipulator, as shown in figure below,



Figure

The inertia tensor matrix for links of the given manipulator is,

$$C_{1_2} = \begin{bmatrix} I_{x_1} & 0 & 0 \\ 0 & I_{y_1} & 0 \\ 0 & 0 & I_{z_1} \end{bmatrix}$$

$$C_{2_2} = \begin{bmatrix} I_{x_2} & 0 & 0 \\ 0 & I_{y_2} & 0 \\ 0 & 0 & I_{z_2} \end{bmatrix}$$

Let,

m_1 – Mass of link-1 (Rotation link)

m_2 – Mass of link-2 (Movable link)

l_1 – Distance of link-1 from centre point O to mass centre

d_2 – Variable distance of link-2 from centre 'O' to mass centre of link '2'.

The equation of motion using Lagrangians-Euler's principle is,

$K.E$ of link-1,

$$K_1 = \frac{1}{2} m_1 l_1^2 \cdot \dot{\theta}_1^2 + \frac{1}{2} I_{z_1} \cdot \dot{\theta}_1^2$$

K.E of link-2,

$$K_2 = \frac{1}{2} m_2 \left[d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right] + \frac{1}{2} I_{z_2} \dot{\theta}_1^2$$

$\therefore$ The total kinetic energy,

$$K = \frac{1}{2}\left[m_1 l_1^2 + I_{z_1} + I_{z_2} + m_2 d_2^2 \right] \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{d}_2^2$$

Potential energy of link-1.

$$U_1 = m_1 g . l_1 \sin\theta_1 + m_1 l_1 . g$$

Potential energy of link-2,

$$U_2 = m_2 g d_2 \sin\theta_1 - m_2 g\, d_{2_{max}}$$

Where, $d_{2_{max}}$ is the maximum extension of joint 2.

∴ The total potential energy,

$$U = (m_1 l_1 + m_2 d_2)\sin\theta_1\, g + m_1 l_1 g + m_2\, d_{2_{max}} g$$

Partial derivatives of kinetic energy and potential energy are,

$$\frac{\partial(k)}{\partial\theta} = \begin{bmatrix} (m_1 l_1^2 + I_{zz_1} + I_{zz_2} + m_2 d_2^2)\dot\theta_1 \\ m_2 \dot d_2 \end{bmatrix}$$

$$\frac{\partial k}{\partial\theta} = \begin{bmatrix} 0 \\ m_2 d_2 \dot\theta_1^2 \end{bmatrix}$$

$$\frac{\partial u}{\partial\theta} = \begin{bmatrix} g(m_1 l_1 + m_2 d_2)\cos\theta_1 \\ m_2 \sin\theta_1 g \end{bmatrix}$$

And,

$$\frac{d}{dt}\left(\frac{\partial k}{\partial\dot\theta}\right) - \frac{\partial k}{\partial\theta} + \frac{\partial u}{\partial\theta} = \tau$$

$$\tau_1 = \left(m_1 l_1^2 + I_{zz_1} + I_{zz_2} + m_2 d_2^2\right)\ddot\theta_1 + 2 m_2 d_2 \dot\theta_1 \dot d_2 + (m_1 l_1 + m_2 d_2)\cos\theta_1 g$$

$$\tau_2 = m_2 \ddot d_2 - m_2 d_2 \dot\theta_1^2 + m_2 \sin\theta_1 g$$

$$M(\theta) = \begin{bmatrix} \left(m_1 l_1^2 + I_{zz_1} + I_{zz_2} + m_2 d_2^2\right) & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V(\theta,\dot\theta) = \begin{bmatrix} 2 m_2 d_2 \dot\theta_1 \dot d_2 \\ -m_2 d_2 \dot\theta_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} (m_1 l_1 + m_2 d_2)\cos\theta_1 g \\ d_2 m_2 \sin\theta_1 g \end{bmatrix}$$

Q24. Derive the expression for the joint torques of a two link planar revolute jointed robotic manipulator using Lagrange Euler formation.

Answer : [May/June-14, Set-3, Q6 | Model Paper-IV, Q2(a)]

Lagrange-Euler formulation of a planar 2 – DOF manipulator with both rotary joints and having unequal links as shown in the figure, is obtained by using direct geometric approach.

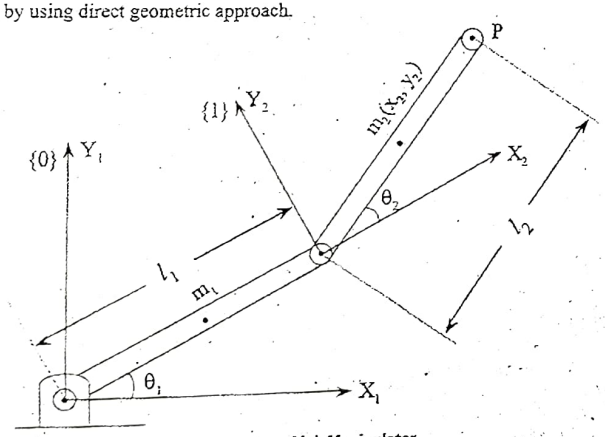


Figure: Two Link Manipulator

Let, {0}, {1}, θ_1, θ_2, l_1, l_2 and m_1, m_2 be the coordinate frames, joint variables, link lengths and masses of the link-1 and 2 respectively.

Kinetic energy of a link is given by,

$$K = \frac{1}{2} mV^2 + \frac{1}{2} I\omega^2$$

Linear velocity (V_1) for link -1 is given by,

$$V_1 = \frac{1}{2} l_1 \dot\theta_1$$

Angular velocity (ω_1) for link -1 is given by,

$$\omega_1 = \dot\theta_1$$

Moment of inertia (I_1) for link -1 is given by,

$$I_1 = \frac{1}{12} m_1 l_1^2$$

$$\therefore K_1 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} I_1 \omega_1^2$$

$$= \frac{1}{8} m_1 l_1^2 \dot\theta_1^2 + \frac{1}{24} m_1 l_1^2 \dot\theta_1^2$$

$$= \frac{1}{6} m_1 l_1^2 \dot\theta_1^2$$

Potential energy (P_1) for link -1 is given by,

$$P_1 = \frac{1}{2} m_1 g l_1 \sin\theta_1$$

The coordinates of centre of mass of link -2 (x_2, y_2) are given as,

$$x_2 = l_1 \cos\theta_1 + \frac{1}{2} l_2 \cos(\theta_1 + \theta_2)$$

$$y_2 = l_1 \sin\theta_1 + \frac{1}{2} l_2 \sin(\theta_1 + \theta_2)$$

Velocity of link-2 is obtained by differentiating x_2 and y_2,

$$\dot x_2 = -l_1 \sin\theta_1 \dot\theta_1 - \frac{1}{2} l_2 \sin(\theta_1 + \theta_2)(\dot\theta_1 + \dot\theta_2)$$

$$\dot y_2 = l_1 \cos\theta_1 + \frac{1}{2} l_2 \cos(\theta_1 + \theta_2)(\dot\theta_1 + \dot\theta_2)$$

Velocity at the end of link -2 is obtained as follows,

$$V_2^2 = \dot x_2^2 + \dot y_2^2$$

$$= l_1^2 S_1^2 \dot\theta_1^2 + \frac{1}{4} l_2^2 S_{12}^2(\dot\theta_1 + \dot\theta_2)^2 + l_1 l_2 S_1 S_{12}(\dot\theta_1^2 + \dot\theta_1 \dot\theta_2) + l_1^2 C_1^2 \dot\theta_1^2 + \frac{1}{4} l_2^2 C_{12}^2(\dot\theta_1 + \dot\theta_2)^2 + l_1 l_2 C_1 C_{12}(\dot\theta_1^2 + \dot\theta_1 \dot\theta_2)$$

$$V_2^2 = l_1^2 \dot\theta_1^2 + \frac{1}{4} l_2^2(\dot\theta_1 + \dot\theta_2)^2 + l_1 l_2 C_2(\dot\theta_1^2 + \dot\theta_1 \dot\theta_2)$$

Where,

$C_i = \cos\theta_i$

$S_i = \sin\theta_i$

$C_{12} = \cos(\theta_1 + \theta_2)$

$S_{12} = \sin(\theta_1 + \theta_2)$.

Kinetic energy of link -2 is given by,

$$K_2 = \frac{1}{2} m_2 V_2^2 + \frac{1}{2} I_2 \omega_2^2$$

And, $\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$

$$I_2 = \frac{1}{12} m_2 l_2^2$$

$$\therefore \quad K_2 = \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + \frac{1}{4} l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)) + \frac{1}{24} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$= \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{6} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)$$

$$= \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{6} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) + \frac{1}{2} m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)$$

Potential energy of link -2 is given by,

$$P_2 = m_2 g l_1 S_1 + \frac{1}{2} m_2 g l_2 S_{12}$$

Lagrangian $L = K - P$

$$= (K_1 + K_2) - (P_1 + P_2)$$

Substituting and simplifying K_1, K_2, P_1 and P_2 values, we get,

$$L = \frac{1}{2}\left(\frac{1}{3} m_1 + m_2\right) l_1^2 \dot{\theta}_1^2 + \frac{1}{6} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) + \frac{1}{2} m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) - \left(\frac{1}{2} m_1 + m_2\right) g$$

$$l_1 S_1 - \frac{1}{2} m_2 g L_2 S_{12}$$

Torque τ_1 at joint 1 is obtained from the following relation,

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{dL}{d\theta_1} \qquad \ldots (1)$$

$$\frac{\partial L}{\partial \theta_1} = -\left(\frac{1}{2} m_1 + m_2\right) g l_1 C_1 - \frac{1}{2} m_2 g l_2 C_{12}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \left(\frac{1}{3} m_1 + m_2\right) l_1^2 \dot{\theta}_1 + \frac{1}{3} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{3} m_2 l_1 l_2 C_2 (2\dot{\theta}_1 + \dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left[\left(\frac{1}{3} m_1 + m_2\right) l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2\right] \ddot{\theta}_1 + m_2 \left[\frac{1}{3} l_2^2 + \frac{1}{2} l_1 l_2 C_2\right] \ddot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 -$$

$$\frac{1}{2} m_2 l_1 l_2 S_2 \dot{\theta}_2^2$$

Substituting $\frac{\partial L}{\partial \theta_1}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)$ in equation (1), we get,

$$\tau_1 = \left[\left(\frac{1}{3} m_1 + m_2\right) l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2\right] \ddot{\theta}_1 + m_2 \left[\frac{1}{3} l_2^2 + \frac{1}{2} l_1 l_2 C_2\right] \ddot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} m_2 l_1 l_2 S_2 \dot{\theta}_2^2 +$$

$$\left(\frac{1}{2} m_1 + m_2\right) g l_1 C_1 + \frac{1}{2} m_2 g l_2 C_{12}$$

Similarly, for joint (2)

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{dL}{d\theta_2}$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} m_2 l_1 l_2 S_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) - \frac{1}{2} m_2 g l_2 C_{12}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{3} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} m_2 l_1 l_2 C_2 \dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \left[\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right] \ddot{\theta}_1 + \frac{1}{3} m_2 l_2^2 \ddot{\theta}_2 - \frac{1}{2} m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$

Substituting $\frac{\partial L}{\partial \theta_2}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)$ in equation (2), we get,

$$\tau_2 = m_2 \left[\frac{1}{3} l_2^2 + \frac{1}{2} l_1 l_2 C_2\right] \ddot{\theta}_1 + \frac{1}{3} m_2 l_2^2 \ddot{\theta}_2 + \frac{1}{2} m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + \frac{1}{2} m_2 g l_2 C_{12}$$

Therefore, the torque equations τ_1 and τ_2 for joints 1 and 2 can be written in generalised form as shown below.

$$\tau_1 = M_{11} \ddot{\theta}_1 + M_{21} \ddot{\theta}_2 + H_1 + G_1$$

$$\tau_2 = M_{21} \ddot{\theta}_1 + M_{22} \ddot{\theta}_2 + H_2 + G_2$$

Where,

M_{ii} – Effective inertia

M_{ij} – Effective coupling inertia

H_i – Centrifugal and coriolis acceleration forces.

$$M_{11} = \left[\left(\frac{1}{3} m_1 + m_2\right) l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2\right]$$

$$M_{22} = \frac{1}{3} m_2 l_2^2$$

$$M_{12} = M_{21} = m_2 \left[\frac{1}{3} l_2^2 + \frac{1}{2} l_1 l_2 C_2\right]$$

$$H_1 = -m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} m_2 l_1 l_2 S_2 \dot{\theta}_2^2$$

$$H_2 = \frac{1}{2} m_2 l_1 l_2 S_2 \dot{\theta}_1^2$$

$$G_1 = \left[\left(\frac{1}{2} m_1 + m_2\right) l_1 C_1 + \frac{1}{2} m_2 l_2 C_{12}\right] g$$

$$G_2 = \frac{1}{2} m_2 l_2 C_{12} g$$

Q25. Explain Newton-Euler formulation for a robot arm.

Answer :

[Nov.-17, set-1, 05, Model Paper-I, 2019]

Newton-Euler Formulation for a Robot Arm

The principles involved in Newton-Euler formulation for a robot arm are Newton's second law and *D' Alembert's* principle. This gives the set of equations of forces acting on the robot manipulator and results in a recursive solution. The kinematic equations such as velocity and acceleration are given by forward recursion, whereas the dynamic equations like forces and moments are given by backward recursion. In this formulation, the position, velocity and acceleration of the manipulator are assumed to be known, while the joint torques are determined by using Newton-Euler dynamic equations.

The following are the motion parameters of the joints involved in robotic manipulator,

(a) Force acting at centre of mass of the link

(b) Angular velocity of the link

(c) Moment of inertia tensor of link

(d) Linear velocity and acceleration of link etc.,

(e) Torque required to move link.

Assume, a rigid link 'n' of the manipulator with single degree of freedom, whose mass distribution depends on the position of centre of mass and inertia tensor. The torques and acceleration of .....wthe link are functions of mass centre and inertia tensor. The geometry of the figure is shown below.

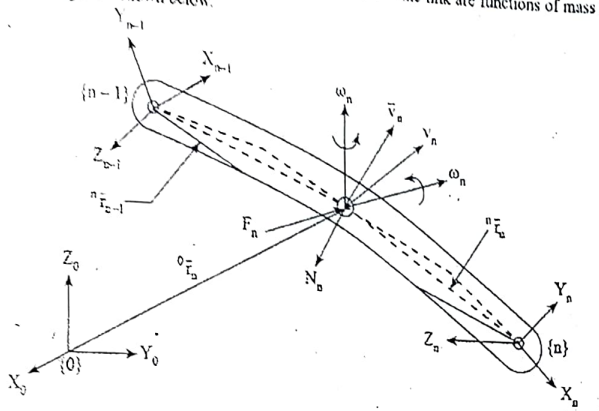


Figure: Geometry of Link 'n' of Manipulator

The two end frames of the link are $\{n\}$ and $\{n-1\}$, while frame $\{0\}$ is the reference. From the figure,

C_n – Centre of mass of link-n

m_n – Mass of link-n

I_i – Inertia tensor of link-n

$\bar{v}_n$ – Linear velocity of centre of mass of link-n

$\bar{v}_n$ – Linear acceleration of centre of mass

ω_n – Angular velocity of link-n

$\dot{\omega}_n$ – Angular acceleration of link-n.

From Newton's second law of motion, the forces acting on the link-n are given by,

$$F_n = m_n \bar{v}_n \quad \ldots (1)$$

Where,

F_n – Force acting on link-n at the centre of mass

From the Euler equation, total moment of the link-n,

$$N = \frac{d}{dt}(I_n \omega_n)$$

$$= I_n \dot{\omega}_n + \omega_n (I_n \omega_n) \quad \ldots (2)$$

Where,

$\omega_n \times I_n \omega_n$ – Gyroscopic torque of the link due to the dependence of inertia tensor (I_n) on its orientation with base frame or reference.

Thus, equations (1) and (2) represents Newton-Euler equations for determining both kinematic equations and dynamic equations of the link-'n'.

Angular velocity,

$^{n-1}\omega_n = 0 \rightarrow$ Prismatic joint

$^{n-1}\omega_n \rightarrow$ Revolute joint.

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Linear velocity,

$V_n = V_{n-1} + \dot{d}_n \rightarrow$ Prismatic joint

$= V_{n-1} + \omega_n \times {}^{n-1}D_n \rightarrow$ Revolute joint.

Acceleration of link $= \omega_n \times (\omega_n \times {}^{n-1}D_n)$

Q26. What are the apparent advantages and disadvantages of the Euler-Lagrange and Newton-Euler formulations?

Answer :

Apparent advantages and disadvantages of Euler-Lagrange formulation,

Advantages of Euler-Lagrange Formulation

1. Euler-Lagrange formulation follows systematic order, which is easy to understand.
2. It gives the equation of motion in a short analytical form consisting inertia matrix, centrifugal matrix and Coriolis forces. This analytical form is useful for control designs.
3. It is effective in flexible link deformations.

Disadvantages of Euler-Lagrange Formulation

1. Euler-Lagrange equation of motion does not have any forward and recurrence motion.
2. Its equation require specification of the velocity at manipulator joints.
3. It gives inaccurate results, due to complex matrices.

Apparent advantages and disadvantages of Newton-Euler formulation

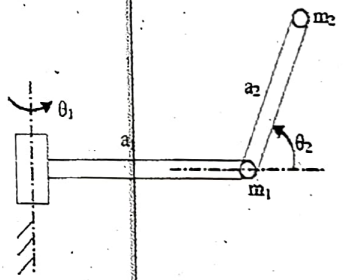
Advantages of Newton-Euler Formulation

1. Newton-Euler formulation computes the recursive method effectively.
2. It can provide solutions concerned to direct dynamics and inverse dynamics.
3. It gives the equation of motion of manipulator joints, end-effector forces and torques.

Disadvantages of Newton-Euler Formulation

1. Newton-Euler formulation of simple physical interpolation cannot be handled easily.
2. It requires software tool to manipulate the robotic parameters such as joint coordinates, joint angles, lengths, kinematic structures etc.
3. It calculates the forces, which do not effect the manipulator.

Q27. Obtain the dynamic equations for the two-link manipulator shown in figure. Assume that whole mass of the link can be considered as a point mass located at the outermost end of each link. The masses are m_1 and m_2 and the link lengths are a_1 and a_2.



Figure

Answer : Nov.-10, Set-3, Q5

The frame assignment and the joint link parameter for the given two-link manipulator is given below,

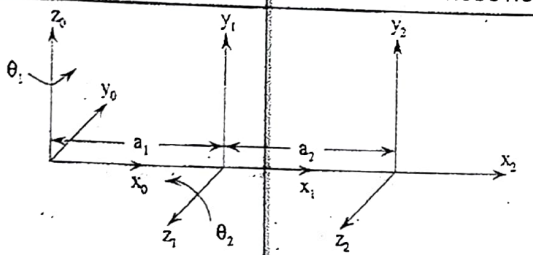


Figure: Frame Assignment for Two-link Manipulator

Joint link parameters for RR non-planar arm are,

Link	Link length	Link Twist	Joint distance	Joint Angle	$C\theta_i$	$S\theta_i$	$C\alpha_i$	$S\alpha_i$
i	a_i	α_i	d_i	θ				
1	a_1	90°	0	θ_1	C_1	S_1	0	1
2	a_2	0	0	θ_2	C_2	S_2	1	0

General transformation matrix,

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & -C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The link transformation matrices are,

$$^0T_1 = \begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation matrix is given by,

$$^0T_2 = {}^0T_1{}^1T_2$$

$$^0T_2 = \begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = {}^0T_1{}^1T_2 = \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 & a_2C_1C_2 + a_1C_1 \\ S_1C_2 & -S_1S_2 & -C_1 & a_2S_1C_2 + a_1S_1 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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For revolute joints (1) and (2), we have,

$$Q_1 = Q_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The inertia tensions I_1 and I_2 for two slender links with point masses m_1 and m_2 is given by,

$$I_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}$$

The coefficients d_{ij} for $i, j = 1, 2$ are calculated using the equations given below,

$$d_{ij} = \begin{cases} {}^0T_{j-1}\,Q_j\,{}^{j-1}T_i & \text{for } j \le i \\ 0 & \text{for } j > i \end{cases}$$

$$d_{11} = {}^0T_0\,Q_1\,{}^0T_1$$

$$= \begin{bmatrix} -S_1 & 0 & C_1 & -a_1S_1 \\ C_1 & 0 & S_1 & a_1C_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$d_{12} = 0 (\because j > i)$$

$$d_{21} = {}^0T_2\,Q_1\,{}^0T_2$$

$$= \begin{bmatrix} -S_1C_2 & S_1S_2 & C_1 & -a_2S_1C_2 - a_1S_1 \\ C_1C_2 & -C_1S_2 & S_1 & -a_2C_1C_2 + a_1C_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$d_{22} = {}^0T_1\,Q_2\,{}^1T_2$$

$$= \begin{bmatrix} -C_1S_2 & -C_1C_2 & 0 & -a_2C_1S_2 \\ -S_1S_2 & -S_1C_2 & 0 & -a_2S_1S_2 \\ C_2 & -S_2 & 0 & a_2C_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The elements of inertia matrix M are calculated by using the formula given below,

$$M_{ij} = \sum_{p=\max(i,j)}^{n} T_r[d_{pj} I_p d_{pi}^T] \text{ for } i, j = 1, 2$$

$$M_{11} = T_r(d_{11} I_1 d_{11}^T) + T_r(d_{21} I_2 d_{21}^T)$$

$$M_{11} = m_1 a_1^2 + (a_2C_2 + a_1)^2 m_2 \text{ and}$$

$$M_{22} = T_r(d_{22} I_2 d_{22}^T) = m_2 L_2^2$$

$$M_{12} = M_{21} = T_r(d_{22} I_2 d_{21}^T) = 0$$

The Coriolis and centrifugal force coefficient are calculated by the formula given below for $i, j, k = 1, 2$

$$h_{ijk} = \sum_{p=\max(i,j,k)}^{n} \left[\frac{\partial (d_{pjk})}{\partial q_r} I_p d_{pi}^T \right]$$

$$h_{111} = 0$$

$$h_{222} = 0$$

$$h_{122} = T_r[{}^0T_1\,Q_2\,{}^1T_1\,I_2 d_{21}^T]$$

$$= -a_2C_1C_2m_2(-a_2S_1C_2 - a_1S_1) - a_2S_1C_2m_2(a_2C_1C_2 + a_1C_1) = 0$$

$$h_{211} = T_r[{}^0T_0\,Q_1\,{}^0T_0\,Q_2\,{}^0T_2\,I_2 d_{22}^T]$$

$$= -a_2C_1S_2m_2(-a_2C_1C_2 - a_1C_1) - a_2S_1S_2m_2(-a_2S_1C_2 - a_1S_1)$$

$$= m_2S_2(a_2^2C_2 + a_1a_2)$$

$$h_{121} = h_{112} = T_r[{}^0T_0\,Q_1\,{}^0T_1\,Q_2\,{}^1T_2\,I_2 d_{22}^T]$$

$$= a_2S_1S_2m_2(-a_2S_1C_2 - a_1S_1) - a_2C_1S_2m_2(a_2C_1C_2 + a_1C_1)$$

$$= -m_2S_2(a_2^2C_2 + a_1a_2)$$

$$h_{221} = h_{212} = T_r[{}^0T_0\,Q_1\,{}^0T_1\,Q_2\,{}^1T_2\,I_2 d_{22}^T] = 0$$

The Coriolis and centrifugal torque terms are calculated by the formula given below,

$$H_i = \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ijk}\dot{\theta}_j\dot{\theta}_k$$

$$H_1 = \sum_{j=1}^{2} \sum_{k=1}^{2} h_{1jk}\dot{\theta}_j\dot{\theta}_k$$

$$= h_{111}\dot{\theta}_1^2 + h_{112}\dot{\theta}_1\dot{\theta}_2 + h_{121}\dot{\theta}_1\dot{\theta}_2 + h_{122}\dot{\theta}_2^2$$

On simplification,

$$H_1 = -2m_2S_2(a_2^2C_2 + a_1a_2)\dot{\theta}_1\dot{\theta}_2$$

Similarly,

$$H_2 = h_{211}\dot{\theta}_1^2 + h_{212}\dot{\theta}_1\dot{\theta}_2 + h_{221}\dot{\theta}_1\dot{\theta}_2 + h_{222}\dot{\theta}_2^2$$

$$H_2 = m_2S_2(a_2^2C_2 + a_1a_2)\dot{\theta}_1^2$$

Therefore,

$$H(\theta, \dot{\theta}) = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} -2m_2S_2(a_2^2C_2 + a_1a_2)\dot{\theta}_1\dot{\theta}_2 \\ m_2S_2(a_2^2C_2 + a_1a_2)\dot{\theta}_1^2 \end{bmatrix}$$

As the mass of the each link is concentrated at the end of the link,

$$^{1}\bar{r}_1 = {}^{2}\bar{r}_2 = [0 \quad 0 \quad 0 \quad 1]^T$$

And, as the gravity is acting in negative z-direction with respect to frame {0},

$$g = [0 \quad 0 \quad -g]$$

Therefore, the gravity loading at joint (1) is given by,

$$G_1 = (m_1 g d_{11}\,{}^{1}\bar{r}_1 + m_2 g d_{21}\,{}^{2}\bar{r}_2) = 0$$

Where, $g = 9.81$ m/s^2

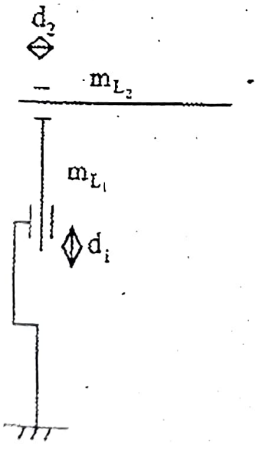
Similarly, at joint-2,

$$G_2 = -m_2 g d_{22}\,{}^{2}\bar{r}_2 = m_2 g a_2 C_2$$

Substituting all these values, the equation of motion in the vector matrix form is obtained as,

$$\therefore \begin{bmatrix}\bar{\tau}_1\\ \bar{\tau}_2\end{bmatrix} = \begin{bmatrix} m_1 a_1^2 + (a_2 C_2 + a_1)^2 m_2 & 0 \\ 0 & m_2 a_2^2 \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} -2m_2 S_2 (a_2^2 C_2 + a_1 a_2)\dot{\theta}_1 \dot{\theta}_2 \\ m_1 S_2 (a_2^2 C_2 + a_1 a_2)\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ m_2 g a_2 C_2 \end{bmatrix}$$

Q28. Obtain equations of motion for the two-link Cartesian arm shown in figure using lagrange formulation. m_{L_1} and m_{L_2} are the masses of the two links, m_{m_1} and m_{m_2} are the masses of rotors of the two joint motors. I_{m_1} and I_{m_2} are the moments of inertia with respect to the axes of the two rotors. It is assumed that the motors are located on the joint axes.



Figure

Answer : [Nov.-11, Set-4, Q5 | Model Paper-III, Q3(b)]

From the above figure of two-link Cartesian arm, the vector generalized coordinates is,

$$q = [d_1 \quad d_2]^T$$

The Jacobian for links (1) and (2) are given as,

$$J_p^{(L_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_p^{(L_2)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

These links do not contribute any angular accelerations.

In order to express the rotor kinetic energy as a function of linear velocity of the rotor centre of mass, the Jacobian is computed as,

$$J_p^{(m_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad J_p^{(m_2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_0^{(m_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad J_0^{(m_2)} = \begin{bmatrix} 0 & K_{r_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Where, K_{r_2} is the gear reduction ratio of motor-2.

And, $Z_1 = [1 \quad 0 \quad 0]^T$

The inertia matrix (B) is given by the summation of the above components relating to single link, single rotors and the total kinetic energy of the manipulator with actuators as given below,

$$B = \begin{bmatrix} m_{L_1} + m_{m_2} + K_{r_1}^2 I_{m_1} + m_{L_2} & 0 \\ 0 & m_{L_2} + K_{r_2}^2 I_{m_2} \end{bmatrix}$$

From the above matrix, it can be noted that, the inertia matrix (B) does not depend on arm configuration.

Also, $C = 0$, since there are no centrifugal and Coriolis components.

The Jacobian matrix considering the gravitational turns is given by,

$$g_0 = [0 \quad 0 \quad -g]^T$$

$$g_1 = (m_{L_1} + m_{m_2} + m_{L_2})g$$

$$g_2 = 0.$$

Neglecting the friction and tip contact forces, the equation of motion can be written as,

$$(m_{L_1} + m_{m_2} + K_{r_1}^2 I_{m_1} + m_{L_2})\ddot{d}_1 + (m_{L_1} + m_{m_2} + m_{L_2})g = \tau_1$$

$$(m_{L_2} + K_{r_2}^2 I_{m_2})\ddot{d}_2 = \tau_2$$

τ_1 and τ_2 are the forces applied to the two joints.

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