# UNIT 3

# MOTION ANALYSIS AND MANIPULATOR KINEMATICS

SIA GROUP

## PART-A
## SHORT QUESTIONS WITH SOLUTIONS

**Q1. Write short notes on robot arm kinematics.**

Model Paper-IV, Q1(e)

**Answer :**

**Robot Arm Kinematics**

The study of geometry of motion of a robot arm with respect to a fixed frame of reference, excluding the forces and moments on the arm, is called robot arm kinematics.

It indicates the displacement of robot in terms of time. It also gives the relation between the joint variables of space, position and orientation of end-effector.

**Q2. What is transformation matrix?**

**Answer :**

**Transformation Matrix**

The position and orientation of robot tool is determined by formulating the body coordinate system with respect to joint axis of the manipulator. Thus, transformation matrix gives the relation between body attached coordinate system and reference coordinate system. It includes,

(i) Rotation matrix
(ii) Translation matrix/Position vector
(iii) Perspective transformation
(iv) Scaling / Stretching.

**Q3. Differentiate between forward and inverse kinematics.**

Model Paper-II, Q1(f)

**Answer :**

| | Forward Kinematics | | Inverse Kinematics |
|---|---|---|---|
| 1. | It is a computation of position and orientation of robot's end effector from joint angles. | 1. | It is a computation of joint angles from position and orientation of robot's end effector. |
| 2. | The input in this method is joint angle vectors. | 2. | The input in this method is position and orientation of end effector. |
| 3. | The position of links are expressed using Cartesian space and joint space. | 3. | The joint angles are determined by geometric, trigonometric or algebraic method. |
| 4. | It determines the actual location of points in a parameterized configuration space. | 4. | It determines the pose of the robot's end effector or tool. |
| 5. | It is a simple method and takes less time in real time control of manipulators. | 5. | It is a complex method and takes more time in real time control of manipulators. |

**Q4. Define D-H convention and state its expression.**

Model Paper-I, Q1(b)

**Answer :**

Denavit-Hartenberg (D-H) convention procedure was introduced for an open chain kinematic manipulator with links and joints. In this chain, each link is provided with right handed orthogonal coordinate system. It is useful to obtain homogeneous transformation matrix for each joint with respect to reference coordinate system. Each transformation '$A_n$' is given as the product of four basic transformations.

i.e., $A_n = Rot_{z,\theta_n} \times Trans_{z,d_n} \times Trans_{x,a_n} \times Rot_{x,\alpha_n}$

$$= \begin{bmatrix} C_{\theta_n} & -S_{\theta_n} & 0 & 0 \\ S_{\theta_n} & C_{\theta_n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{\alpha_n} & -S_{\alpha_n} & 0 \\ 0 & S_{\alpha_n} & C_{\alpha_n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\theta_n} & -S_{\theta_n} C_{\alpha_n} & S_{\theta_n} S_{\alpha_n} & a_n C_{\theta_n} \\ S_{\theta_n} & C_{\alpha_n} C_{\alpha_n} & -C_{\theta_n} S_{\alpha_n} & a_n S_{\theta_n} \\ 0 & S_{\alpha_n} & C_{\alpha_n} & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$\theta_n$, $a_n$, $d_n$, $\alpha_n$ – Parameters of link '$n$' and joint '$n$'.

**Q5. Describe kinematic modeling of manipulator.**

**Answer :**

The geometrical motion of a manipulator can be defined with the help of analytical method of kinematic model, which is a function of time, with respect to a fixed reference. The kinematic model is given by the relation between joint variables, position and orientation of end-effector. It is used to control the motion of end-effector. The two types of kinematic modeling problem are,

1. **First Problem**

When joint variables are known, position and orientation of end-effector are to be determined for a manipulator with '$n$' degrees of freedom. It is called direct kinematics.

2. **Second Problem**

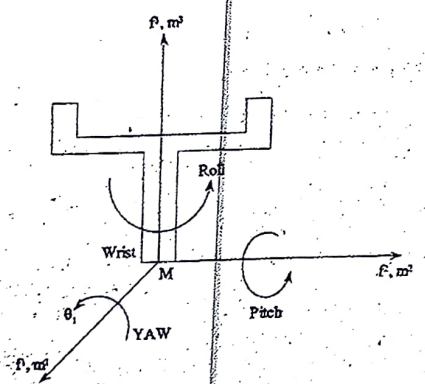
When the position and orientation of end-effector are known, joint-variables are required to be determined, by considering inertial frame of reference. This is called inverse kinematics.

**Q6. Explain Yaw-Pitch-Roll (YPR) transformation with an example.**

**Answer :**

**Yaw-Pitch-Roll (YPR) Transformation**

Consider the figure below for transformation matrix that rotates $M$ by $\theta_1$ about $f^1$, then by $\theta_2$ about $f^2$, and then by $\theta_3$ about $f^3$.




Figure

Using the composite rotation algorithm,

1. $R = I$ (Identity matrix)

2. $R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$



When co-ordinate frames are attached to robot joints, the origins of the frames are not identical, which need to do transformation.

$$R_2(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_3(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

YPR transformation contains the fundamental rotations $\{R_1, R_2, R_3\}$.

For $R_1$ evaluation,

$$\text{YPR}(\theta) = \begin{bmatrix} c_2c_3 & s_1s_2c_3 - c_1s_3 & c_1s_2c_3 - s_1s_3 \\ c_2s_3 & s_1s_2s_3 + c_1c_3 & c_1s_2s_3 - s_1c_3 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix}$$

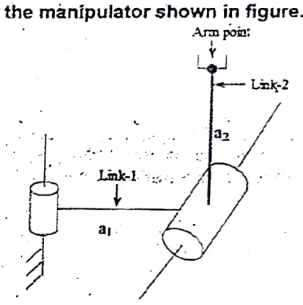
When, $\theta_2 = \theta_3 = 0$

Where,

$c = \cos\theta$

$s = \sin\theta$

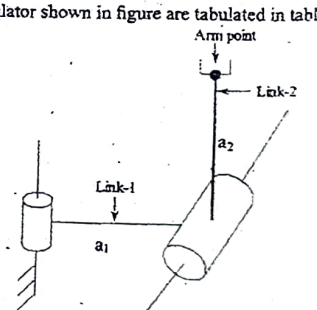
**Q7. Obtain the D-H parameters for the manipulator shown in figure.**




Figure

**Answer :**

The D-H parameters for the manipulator shown in figure are tabulated in table.




Figure

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 90° | 0 | $\theta_1$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2$ |

Table: D-H Parameters

**Q8. For the following rotation matrix determine the axis of rotation and the angle of rotation about the same axis.**

$$\begin{bmatrix} \sqrt{3}/2 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & \sqrt{3}/2 \end{bmatrix}$$

**Answer :**

Given that, [Nov.-10, Set-3, Q3(a) | Model Paper-III, Q1(a)]

$$\text{Rotation matrix, } R = \begin{bmatrix} \sqrt{3}/2 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & \sqrt{3}/2 \end{bmatrix} \quad \ldots (1)$$

The above matrix is in the form of,

$$R_{(y,\theta)} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \ldots (2)$$

This represents the rotation matrix to perform rotation about the y-axis through an angle θ.

∴ From equations (1) and (2),

$$\cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = 0.5$$

$$\theta = 30°$$

∴ Axis of rotation is y-axis and angle of rotation is 30°.

**Q9. Write the homogeneous transformation matrix for a rotation of 90° about the z-axis, followed by a rotation of 90° about the x-axis, followed by a translation of (3, 7, 9).**

**Answer :**

The homogeneous transformation matrix (T) is to perform rotation of 90° about Z-axis, followed by rotation of 90° about X-axis, followed by translation of (3, 7, 9) is given as,

$$T = Trans(3, 7, 9) \times Rot(x, 90°) \times Rot(z, 90°)$$

$$= \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad T = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 7 \\ 1 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

∴ The above matrix is the homogeneous transformation matrix.

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**Q10. A coordinate frame {B} is located initially coincident with a coordinate frame {A}. If the frame {B} is rotated through 30 degrees about $Z_B$ and 60 degrees about current $Y_B$, find the rotation matrix that will describe a vector of frame {B} in frame {A}.**

**Answer :** Nov.-11, Set-1, Q3(b)

Given that,

θ = Angle made by frame {B} about Z-axis = 30°

ϕ = Angle made by frame {B} about Y-axis = 60°

Also, ${}^A_B R = R_{Z,\theta} \cdot R_{Y,\phi}$

${}^A_B R = {}^A_B R$

∴ Substitute θ = 30° and ϕ = 60° in the above equation,

$${}^A_B R = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos 60 & 0 & \sin 60 \\ 0 & 1 & 0 \\ -\sin 60° & 0 & \cos 60° \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & 0.5 \end{bmatrix}$$

$$\therefore {}^A_B R = \begin{bmatrix} 0.433 & -0.5 & 0.749 \\ 0.25 & 0.866 & 0.433 \\ -0.866 & 0 & 0.5 \end{bmatrix}$$

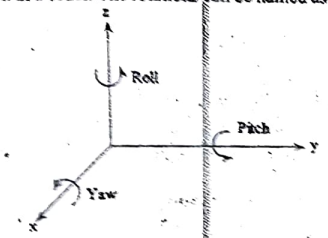
# PART-B
## ESSAY QUESTIONS WITH SOLUTIONS

## 3.1 HOMOGENEOUS TRANSFORMATIONS AS APPLICABLE TO ROTATION AND TRANSLATION – PROBLEMS

**Q11. What are X, Y, Z fixed angles?**

**Answer :**

*X, Y, Z* fixed angles are concerned with robot kinematics for establishing various co-ordinate systems to represent the positions and orientations of rigid objects. A rotation matrix can be expressed as a product of successive rotations about the principal co-ordinate axes *X, Y* and *Z* taken in a order. The rotations can be named as roll, pitch and yaw angles shown in figure.




Let, x-axis represents yaw angle $\psi$

y-axis represents pitch angle $\theta$

z-axis represents roll-angle $\phi$

The resulting matrix can be written as,

$$R_0 = R_{Z,\phi}\, R_{Y,\theta}\, R_{X,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

On simplifying,

$$= \begin{bmatrix} c_\phi c_\theta & (-s_\phi c_\psi + c_\phi s_\theta s_\psi) & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & (c_\phi c_\psi + s_\phi s_\theta s_\psi) & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

**Q12. Explain about equivalent axis and angle representation.**

May/June-14, Set-2, 03(b)

**Answer :**

For rigid bodies, to represent the orientation it requires three rotational degree of freedom. Thus, the arbitrary rotation of an object is represented by three independent quantities. They are as follows.

1. Equivalent axis and angle representation
2. Euler angles representation
3. Roll-pitch-yaw representation

**Equivalent Axis and Angle Representation**

If the orientation of a body is through the single rotation about an arbitrary axis, then it is known as equivalent axis or angle representation. Consider a rotation matrix $(R)$, represented by single rotation about an arbitrary axis by the corresponding angle in space,

i.e., $(R) = R_{k,\theta}$

Where,

'k' is unit vector for axis of rotation i.e., equivalent axis.

'$\theta$' is angle of rotation about 'k' i.e., equivalent angle.



For rotation matrix,

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The equivalent axis is,

$$k = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

The equivalent angle is,

$$\theta = \cos^{-1}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

**Q13. Define translation transformation and explain how the co-ordinate of the vector changes.**

**Answer :**

**Three-dimensional (3D) Transformation**

3D transformations are similar to 2D transformation, both definition and derivation. The 3D translation, scaling and rotation transformation matrix is formed as follows.

**Translation**

In this case we translate a point $P(x, y, z)$ by $(dx, dy, dz)$ to point $P'(x', y', z')$. This can be expressed in matrix form as follows,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

**Scaling**

If $S$ is the scaling coefficient matrix, then the scaling transformation in 3D is,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

**Rotations**

The rotation of an object could be about any axes, the rotational transformations about all of the three axes are as follows,

**1. About z**

When it is required to rotate about the z-axis, the z co-ordinate of a point $P(x, y, z)$ after rotation will not change because the rotation takes place in the x-y plane. Therefore, the new $(x, y)$ will be same as those obtained in the case of 2D rotations. Suppose, location of initial point is $P(x, y, z)$ then the location of new point $P'(x', y', z')$ as follows,

$$z' = z$$

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

In matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Similarly, obtain the equations for rotation about x and y-axis.

**2. About x**

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

**3. About y**

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

In general, $P' = [R]\, P$

Where $[R]$ is the rotational matrix i.e. [illegible]

**Q14. Explain the homogeneous transformation applicable to rotation.**

**Answer :** [illegible]

It is a general method for solving the kinematics of a robot manipulator with many joints. It is another single matrix that combines the effect of translation and rotation.

The rotation transformation operates on vector coordinates and performs rotation about a given axis of the reference coordinate system. The reference coordinate system is given as follows.

(a) Rotation '$\alpha$' degrees about the x-axis.

$$\text{Rot}(x, \alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation '$\alpha$' degrees about the y-axis.

$$\text{Rot}(y, \alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Rotation '$\alpha$' degrees about the z-axis.

$$\text{Rot}(z, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The general form of homogenous transformation can be partitioned into sub-matrix in tabular form.

| Rotation matrix (3×3) | Position vector (3×1) |
|---|---|
| Perspective transform (1×3) | Scaling factor (1×1) |

The (3 × 3) matrix is for rotation, (3 × 1) for translation and other two sub-matrix for perspective transform and scaling factor. Vector and position nomenclature for homogenous transformation matrix,

$$H = \left[\begin{array}{ccc|c} h_x & o_x & a_x & p_x \\ h_y & o_y & a_y & p_y \\ h_z & o_z & a_z & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Homogenous transformation is based on mapping and N-dimensional space into $(N + 1)$ dimensional space i.e., one more coordinate is added to represent the position of a point.

**Example**

A 3-dimensional space point has coordinates $(x, y, z)$, is represented by vector $(x, y, z, w)$ in homogenous transformations in which '$w$' is a dummy coordinate that on normalization gives $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1\right)$. This additional '1' serves as a tool to accomplish the addition of matrices required in translation by matrix multiplication.

**Q15. State and prove the properties of a rotation matrix.**

{Nov.-11, Set-3, Q3(a) | Nov.-10, Set-2, Q3(a) | Model Paper-II, Q3(b)}

**Answer :**

The following are the properties of a rotation matrix,

(i) Each row vector of the rotation matrix as an axis unit vector representation of the reference frame which is expressed in terms of the unit vectors of the rotated axis of $OXYZ$ coordinate system, whereas each column vector of the rotation matrix is a unit vector representation of the rotated axis, which is expressed in terms of axis unit vectors of the reference frame.

(ii) A matrix is said to be a rotation matrix when it is orthogonal and its determinant is unity. The determinant of orthogonal matrix is ± 1.
Where,
+ 1 = Right-hand coordinate system and
− 1 = Left-hand coordinate system.

(iii) Magnitude of each row and column vector must be equal to 1, as each row and column is a representation of unit vector.

(iv) The dot product of each column with other column equals to zero, as each column vector is a orthonormal vectors representation. Similarly, the dot product of each row with other row equals to zero.

(v) The inverse of a rotation matrix is the transpose of rotation matrix.
i.e., $R^{-1} = R^T$

**Case-I :** $R^{-1} = R^T$

Consider rotation matrix for the rotation of an angle '$\alpha$' about Z-axis.

$$R_{(z,\alpha)} = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where,

$C\alpha = \cos\alpha$

$S\alpha = \sin\alpha$

Now,

$$\text{R.H.S, } R^T = \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{L.H.S, } R^{-1} = \frac{1}{\det R}\ [\text{Adjoint of } R]$$

Where,

$$\det R = C\alpha\,[C\alpha(1) - 0(0)] - (-S\alpha)[S\alpha(1) - 0(0)] + 0\,[S\alpha(0) - C\alpha(0)]$$
$$= C^2\alpha + S^2\alpha$$
$$= \cos^2\alpha + \sin^2\alpha$$

$\det R = 1$ and

Now, adjoint of matrix $R$

$$R = \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & S\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine, the co-factor of each element of the $R$ matrix

Co-factor of $C\alpha = C\alpha(1) - 0(0) = C\alpha$

Co-factor of $-S\alpha = S\alpha(1) - 0(0) = S\alpha$

Co-factor of $0 = S\alpha(0) - C\alpha(0) = 0$

Co-factor of $S\alpha = -S\alpha(1) - 0(0) = -S\alpha$

Co-factor of $C\alpha = C\alpha(1) - 0(0) = C\alpha$

Co-factor of $0 = C\alpha(0) - (-S\alpha(0)) = 0$

Co-factor of $0 = -S\alpha(0) - 0(C\alpha) = 0$

Co-factor of $0 = C\alpha(0) - 0(S\alpha) = 0$

Co-factor of $1 = C\alpha(C\alpha) + S\alpha(S\alpha)$
$= C^2\alpha + S^2\alpha = \cos^2\alpha + \sin^2\alpha = 1$

Thus,

$$\text{Adj of } R = \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1} = \det R.\ \text{Adj}[R]$$

Now,

$$R^{-1} = \frac{1}{1}\begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \text{R.H.S}$$

$\therefore$ L.H.S = R.H.S

i.e., $R^{-1} = R^T$





**Case-II**

The product of rotation matrix and transpose of rotation matrix equals to identity matrix.

i.e.,

$$RR^T = I_3$$
$$(\text{L.H.S}) = (\text{R.H.S})$$

Where,

$I_3 = 3 \times 3$ identity matrix.

**Proof**

Consider rotation matrix for the rotation of an angle '$\alpha$' about Z-axis.

$$R_{(z,\alpha)} = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$\text{L.H.S} = RR^T$$
$$= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\alpha & S\alpha & 0 \\ -S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} C\alpha(C\alpha) - S\alpha(-S\alpha) + 0(0) & C\alpha(S\alpha) - S\alpha(C\alpha) + 0(0) & C\alpha(0) - S\alpha(0) + 0(1) \\ S\alpha(C\alpha) + C\alpha(-S\alpha) + 0(0) & S\alpha(S\alpha) + C\alpha(C\alpha) + 0(0) & S\alpha(0) + C\alpha(0) + 0(1) \\ 0(C\alpha) + 0(-S\alpha) + 1(0) & 0(S\alpha) + 0(C\alpha) + 1(0) & 0(0) + 0(0) + 1(1) \end{bmatrix}$$
$$= \begin{bmatrix} C^2\alpha + S^2\alpha & C\alpha S\alpha - S\alpha C\alpha & 0 \\ S\alpha C\alpha - C\alpha S\alpha & S^2\alpha + C^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\because\ C^2\alpha + S^2\alpha = 1)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$
$$= \text{R.H.S}$$

Hence, $RR^T = I_3$

**Q16. Show that the rotation matrix about the y-axis is unitary.**

**Answer :**

To show that the rotation matrix about the $y$-axis is unitary. Consider a rotation matrix for the rotation of angle '$\alpha$' about $y$-axis and for unitary condition.

**Rotation about Y-axis**

$$\text{Rot } (y, \alpha)^{-1} = \text{Rot } (y, \alpha)^T$$

**Proof**

Consider rotation matrix for the rotation of angle '$\alpha$' about $y$-axis.

$$R_y = \begin{bmatrix} C\alpha & 0 & S\alpha \\ 0 & 1 & 0 \\ -S\alpha & 0 & C\alpha \end{bmatrix}$$

Now,

$$R_y^T = \begin{bmatrix} C\alpha & 0 & -S\alpha \\ 0 & 1 & 0 \\ S\alpha & 0 & C\alpha \end{bmatrix} \quad \ldots (1)$$

$$R_y^{-1} = \frac{1}{\det R_y}[Adj\ R_y]$$

$$\det R_y = |R_y| = C\alpha\,[C\alpha - 0] - 0 + S\alpha\,[0 - (-S\alpha)]$$

$$= C^2\alpha + S^2\alpha$$

$$\det R_y = 1$$

$$Adj\ R_y = Adj\begin{bmatrix} C\alpha & 0 & S\alpha \\ 0 & 1 & 0 \\ -S\alpha & 0 & C\alpha \end{bmatrix}$$

Determine the co-factor of each element of the $R_y$ matrix,

Co-factor of $C\alpha = +\,[C\alpha\,(1) - 0] = C\alpha$

Co-factor of $0 = -\,[0 - 0] = 0$

Co-factor of $S\alpha = +\,[0 - (S\alpha)1] = S\alpha$

Co-factor of $0 = -\,[0 - 0] = 0$

Co-factor of $1 = +\,[C^2\alpha + S^2\alpha] = 1$

Co-factor of $0 = -\,[0 - 0] = 0$

Co-factor of $-S\alpha = +\,[0 - S\alpha\,(1)] = -S\alpha$

Co-factor of $0 = -\,[0 - 0] = 0$

Co-factor of $C\alpha = +\,[C\alpha - 0] = C\alpha$.

$$\text{Co-factor matrix, } R_y = \begin{bmatrix} C\alpha & 0 & S\alpha \\ 0 & 1 & 0 \\ -S\alpha & 0 & C\alpha \end{bmatrix}$$

$$\text{Thus, } Adj \text{ of } R_y = \begin{bmatrix} C\alpha & 0 & -S\alpha \\ 0 & 1 & 0 \\ S\alpha & 0 & C\alpha \end{bmatrix}$$

Now, inverse matrix, $R_y^{-1} = \frac{1}{|R_y|}(Adj\ R_y)$

$$R_y^{-1} = \frac{1}{1}\begin{bmatrix} C\alpha & 0 & -S\alpha \\ 0 & 1 & 0 \\ S\alpha & 0 & C\alpha \end{bmatrix} \quad \ldots (2)$$

Compare equations (1) and (2),

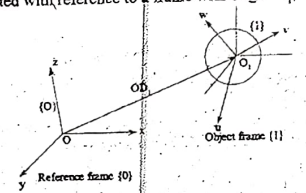
$$R_y^T = R_y^{-1}$$

i.e., Inverse of the rotation matrix about the $y$-axis is the same as its transpose.

**Q17. Show that a rotation matrix is an orthogonal matrix.** Nov.-11, Set-2, Q3(a)

**Answer :**

Location of an object in 3D space is possible by description of its position and orientation with respect to a fixed frame of reference. The object can be represented with reference to a frame with origin $O_1$, it as shown in figure.




Figure





Location of links found out by assigning each link a coordinate frame, right from the base to the end-effector. For an end-effector the coordinate frame is assigned with the help of normal, sliding and approach vectors. The approach vector ($\hat{a}$) is at $z$-axis and it is the direction, where end effector approaches the target. Orientation or sliding vector ($\hat{o}$) is at $y$-axis and the direction of the sliding vector. It is also known as orientation vector. Normal vector ($\hat{n}$) is at $x$-axis that is in orthogonal direction to the approach and sliding vectors.

Transformation matrix ($T$) for the end effector with respect to the frame $\{n\ o\ a\}$ is given by,

$$T = \begin{bmatrix} n_x & o_x & a_x & d_x \\ n_y & o_y & a_y & d_y \\ n_z & o_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & o & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where, $(d_x, d_y, d_z)$ represent the translation of end effector frame with respect to the reference frame while the vectors ($n$, $o$, $a$) give the orientation of end-effector.

End effector rotation matrix ($R$) is given below,

$$R = \begin{bmatrix} \hat{x}.\hat{u} & \hat{x}.\hat{v} & \hat{x}.\hat{w} \\ \hat{y}.\hat{u} & \hat{y}.\hat{v} & \hat{y}.\hat{w} \\ \hat{z}.\hat{u} & \hat{z}.\hat{v} & \hat{z}.\hat{w} \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

Where, the vectors $n$, $o$ and $a$ are in mutually perpendicular directions and the rotation matrix, $R$ is an orthogonal transformation.

The scalar dot product of two different columns is always zero.

$$n.o = o.a = a.n = 0$$

The scalar dot product of a column with itself is unity.

$$n.n = o.o = a.a = 1 \implies |n| = |o| = |a| = 1$$

The vector product between two different columns gives the third column in a cyclic order.

$$n \times o = a,\ o \times a = n,\ a \times n = o$$

The determinant of the rotation matrix is unity,

$$\begin{vmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{vmatrix} = 1$$

**Q18. Define and explain about inverse homogeneous transformations.**

**Answer :**

**Inverse Homogeneous Transformations**

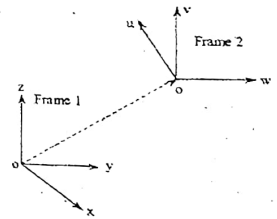



Figure: 3D Space with Two Frames

An inverse homogeneous transformation is defined by relationship among the two frames that can be found by a series of translation and rotation motion.

Figure shows two orthogonal reference frames. The frame-1 i.e., $oxyz$ is considered as a fixed or reference frame while frame-2 i.e., $ouvw$ is considered as movable frame. If the homogeneous transformation matrix is projected from frame-2 to frame-1, then it is expressed by,

$$P_{xyz} = HP_{uvw}$$

$$P_{uvw} = H^{-1}P_{xyz}$$

The inverse homogeneous transformation is found out by first obtaining homogeneous transformation matrix i.e., '$H$'. The general form for the homogeneous transformation matrix is given by,

$$H_{4\times4} = \left(\begin{array}{c|c} \text{Rotation matrix } (3\times3) & \text{Position vector } (3\times1) \\ \hline \text{Perspective transformation } (1\times3) & \text{Scaling factor } (1\times1) \end{array}\right)$$

$$H = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where, subscript '$x$' represents normal vector, subscript '$y$' represents orientation vector and subscript '$z$' represents approach vector.

Let the transformation matrix '$H$' is rotated with an angle '$\alpha$' about $x$-axis i.e.,

$$H(x,\alpha) = \left(\begin{array}{c|c} R_{x(3\times3)} & P_{(3\times1)} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right)$$

$$H(x,\alpha) = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & \cos\alpha & -\sin\alpha & P_y \\ 0 & \sin\alpha & \cos\alpha & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

The inverse of transformation matrix, $H(x, \alpha)$ is given by,

$$H^{-1}(x,\alpha) = \left(\begin{array}{c|c} R^T_{x(3\times3)} & -R^T_{x(3\times1)} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right) \quad \ldots (1)$$

The transpose of matrix $R$ is written by changing the rows into columns of matrix $R$.

$$R^T_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \quad \ldots (2)$$

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Now,

$$R^T_x P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$R^T_x P = \begin{bmatrix} P_x + 0 + 0 \\ 0 + P_y\cos\alpha + P_z\sin\alpha \\ 0 - P_y\sin\alpha + P_z\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} P_x \\ P_y\cos\alpha + P_z\sin\alpha \\ -P_y\sin\alpha + P_z\cos\alpha \end{bmatrix} \quad \ldots (3)$$

Substituting equations (2) and (3) in equation (1),

$$H^{-1}_{(x,\alpha)} = \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & \cos\alpha & \sin\alpha & -(P_y\cos\alpha + P_z\sin\alpha) \\ 0 & -\sin\alpha & \cos\alpha & (P_y\sin\alpha - P_z\cos\alpha) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is known as inverse homogeneous transformation matrix rotated with an angle '$\alpha$' about $x$-axis.
If the transformation matrix '$H$' is rotated with an angle '$\alpha$' about $y$-axis then,

$$H(y,\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & P_x \\ 0 & 1 & 0 & P_y \\ -\sin\alpha & 0 & \cos\alpha & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H(y,\alpha) = \left(\begin{array}{c|c} R_{y(3\times3)} & P_{(3\times1)} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right)$$

Where, $$R_y = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

The inverse of $H(y, \alpha)$ is given by,

$$H^{-1}(y,\alpha) = \left(\begin{array}{c|c} R^T_{y(3\times3)} & -R^T_y P_{(3\times1)} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right) \quad \ldots (4)$$

To get $R^T_y$, change rows into columns of matrix $R_y$.

$$R^T_y = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \ldots (5)$$

$$R_z^T P = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$R_y^T P = \begin{bmatrix} P_x \cos\alpha - P_z \sin\alpha \\ P_y \\ P_x \sin\alpha + P_z \cos\alpha \end{bmatrix} \quad \ldots (6)$$

Substituting equations (5) and (6) in equation (4),

$$H^{-1}(y, x) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha & -(P_x \cos\alpha - P_z \sin\alpha) \\ 0 & 1 & 0 & -P_y \\ \sin\alpha & 0 & \cos\alpha & -(P_x \sin\alpha + P_z \cos\alpha) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is inverse homogenous transformation matrix with an angle '$\alpha$' about y-axis.

Similarly, the inverse homogeneous transformation matrix rotated about z-axis is given as,

$$H^{-1}(z, \alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & -(P_x \cos\alpha - P_y \sin\alpha) \\ -\sin\alpha & \cos\alpha & 0 & (P_x \sin\alpha - P_y \cos\alpha) \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Q19. Draw and explain the composite homogeneous transformation algorithm.**

**Answer :**

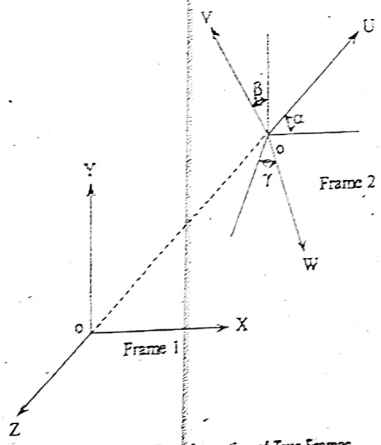



**Figure: Composite Transformation of Two Frames**

A composite homogeneous transformation involves rotation and translation of each frame from its previous frame as shown in figure. Usually, the task done by a robot is described with respect to base frame i.e., fixed repeatable reference frame *OXYZ*.

If the task to be performed is somewhere else in space, then it is necessary to use composite homogeneous transformation as it co-relates the new coordinate system with the fixed coordinate system i.e., base frame. The following rules have to be followed in evaluating the composite homogeneous transformation matrix,

1. Initially, as both the fixed and movable coordinate systems (i.e., frame 1 and 2) are coincident, so, the homogeneous transformation matrix is an identity matrix of order 4 × 4.
2. If the rotating co-ordinate '*OUVW*' is rotated or translated about principle axis of '*OXYZ*', then *pre-multiply* previous rotation matrix (resultant) with an appropriate present rotation (or translation) matrix.

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3. If rotating co-ordinate '*OUVW*' is rotated or translated about its own principle axes, then *post-multiply* the previous matrix (resultant) with an appropriate present rotation matrix.

Example for composite homogeneous transformation.

$$H = H(y, \beta) I_z H(A, \alpha), H(B, d)$$

$$H = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & d\cos\alpha \\ 0 & \sin\alpha & \cos\alpha & d\sin\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & d\cos\alpha \\ 0 & \sin\alpha & \cos\alpha & d\sin\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad H = \begin{bmatrix} \cos\beta & \sin\alpha\sin\beta & \cos\alpha\sin\beta & d\sin\alpha\sin\beta \\ 0 & \cos\alpha & -\sin\alpha & d\cos\alpha \\ -\sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta & d\sin\alpha\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

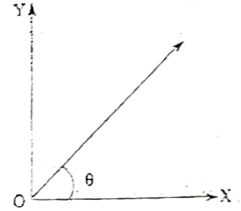
**Q20. What is the rotation matrix for a rotation of ϕ-angle about OX axis, followed by a rotation of θ about the OZ axis, followed by a rotation of θ about the OY axis. Also, find another sequence of rotations that is different from the above problem, but which results in the same rotation matrix?**

**Answer :**

(a) Rotation matrix is any matrix that acts as a rotation in space.

For example, the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

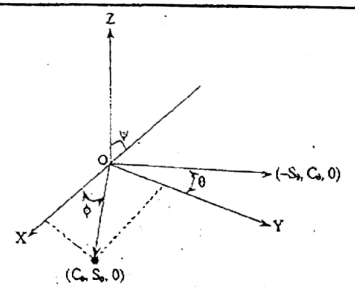
Rotates vectors in *XY* plane in counter clockwise by an angle of θ.




**Figure**

The expressions for rotations about frame axes can be derived. From the figure below, it is seen that the rotation by an angle ϕ about the *OX* axis is described by,

$$R_{x(\phi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$




**Figure**

It can be shown that the rotation of a frame by an angle ψ about the z-axis is described by,

$$R_{z(\psi)} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, the rotation of a frame by an angle θ about the y-axis is described by,

$$R_{y(\theta)} = \begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

(b) A(3 × 3) rotation matrix is the transformation matrix, which operates on a position vector. Basic rotation matrices can be multiplied together to represent a sequence of finite rotations about the principle axes of the *OXYZ* co-ordinate system. To develop a rotation matrix representing, a rotation of ψ angle about the *OX* axis followed by a rotation of θ angle about the *OZ* axis followed by a rotation of ϕ angle about the *OY* axis, the resultant rotation matrix representing these rotation is,

$$R = R_{y(\phi)} R_{z(\theta)} R_{x(\psi)}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi.\cos\theta & \sin\phi.\sin\theta - \cos\phi.\sin\theta.\cos\psi & \cos\phi.\sin\theta.\sin\psi + \sin\phi\cos\psi \\ \sin\theta & \cos\theta.\cos\psi & -\cos\theta.\sin\psi \\ -\sin\phi.\cos\theta & \sin\phi.\sin\phi.\cos\psi + \cos\phi.\sin\psi & \cos\phi.\cos\psi - \sin\phi.\sin\theta.\sin\psi \end{bmatrix}$$

The resultant rotation matrix is,

$$R = R_{x,\psi} \times R_{z,\theta} \times R_{y,\phi}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta.\cos\phi & -\sin\theta & \cos\theta.\sin\phi \\ \cos\psi.\sin\theta.\cos\phi + \sin\psi.\sin\phi & \cos\psi.\cos\theta & \cos\psi.\sin\theta\sin\phi - \sin\theta.\cos\phi \\ \sin\psi.\sin\theta.\cos\phi - \cos\psi.\sin\phi & \sin\psi.\cos\theta & \sin\psi.\sin\theta.\sin\phi + \cos\psi.\cos\phi \end{bmatrix}$$

The resultant matrix is the same for both cases, but the sequence of rotation is different from the one that generates in previous case.

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**Q21. Determine a composite rotation matrix for the following sequence of rotations,**

(i) Rotation of angle α about X-axis

(ii) Rotation of angle β about Y-axis

(iii) Rotation of angle γ about Z-axis.

**Answer :**

[Nov.-10, Set-2, Q3(b) ; Model Paper-I, Q2(a)]

Rotation matrices representing a rotation of an angle are described by,

(i) **Rotation of Angle α about X-axis**

$$R(x, \alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

(ii) **Rotation of Angle β about Y-axis**

$$R_{(y, \beta)} = \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix}$$

(iii) **Rotation of Angle γ about Z-axis**

$$R_{(z, \gamma)} = \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where,

$C\alpha = \cos\alpha$, $S\alpha = \sin\alpha$

$C\beta = \cos\beta$, $S\beta = \sin\beta$

$C\gamma = \cos\gamma$, $S\gamma = \sin\gamma$

The composite rotation matrix ($R$) following the sequence of rotations can be obtained by,

$$R = R_{(x, \alpha)} R_{(y, \beta)} R_{(z, \gamma)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta(C\gamma)+0(S\gamma)+S\beta(0) & C\beta(-S\gamma)+0(C\gamma)+S\beta(0) & C\beta(0)+0(0)+S\beta(1) \\ 0(C\gamma)+1(S\gamma)+0(0) & 0(-S\gamma)+1(C\gamma)+0(0) & 0(0)+1(0)+0(1) \\ -S\beta(C\gamma)+0(S\gamma)+C\beta(0) & -S\beta(-S\gamma)+0(C\gamma)+C\beta(0) & -S\beta(0)+0(0)+C\beta(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta\,C\gamma & -C\beta\,S\gamma & S\beta \\ S\gamma & C\gamma & 0 \\ -S\beta\,C\gamma & S\beta\,S\gamma & C\beta \end{bmatrix}$$

$$= \begin{bmatrix} 1(C\beta\,C\gamma)+0(S\gamma)+0(-S\beta\,C\gamma) & 1(-C\beta\,S\gamma)+0(C\gamma)+0(+S\beta\,S\gamma) & 1(S\beta)+0(0)+0(C\beta) \\ 0(C\beta\,C\gamma)+C\alpha(S\gamma)-S\alpha(-S\beta\,C\gamma) & 0(-C\beta\,S\gamma)+C\alpha(C\gamma)-S\alpha(+S\beta\,S\gamma) & 0(S\beta)+C\alpha(0)-S\alpha(C\beta) \\ 0(C\beta\,C\gamma)+S\alpha(S\gamma)+C\alpha(-S\beta\,C\gamma) & 0(-C\beta\,S\gamma)+S\alpha(C\gamma)+C\alpha(+S\beta\,S\gamma) & 0(S\beta)+S\alpha(0)+C\alpha(C\beta) \end{bmatrix}$$

$$R = \begin{bmatrix} C\beta\,C\gamma & -C\beta\,S\gamma & S\beta \\ C\alpha S\gamma + S\alpha\,S\beta\,C\gamma & C\alpha\,C\gamma - S\alpha\,S\beta\,S\gamma & -S\alpha\,C\beta \\ S\alpha\,S\gamma - C\alpha\,S\beta\,C\gamma & S\alpha\,C\gamma + C\alpha\,S\beta\,S\gamma & C\alpha\,C\beta \end{bmatrix}$$

**Q22. Obtain the rotation matrix corresponding to the set of Euler angles with respect to fixed XZX axes.**

**Answer :** Nov.-10, Set-1, Q3(b)

**Euler Angles**

A method to specify a rotation matrix in terms of three independent quantities is known as "Euler Angles". Figure shows the fixed coordinate frame $x_0$, $y_0$ and $z_0$ and the rotated frame $OXYZ$ by angles ($\alpha$, $\beta$, $\gamma$) known as "Euler Angles".

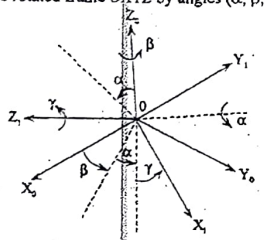



**Figure: Euler Angle Notation**

**Sequence of Rotations**

Rotation matrices must be multiplied together to represent a sequence of rotations about principle axes of the $OXYZ$ coordinate system. To develop a rotation matrix, follow the specified sequence of rotations,

(i) Rotation of angle '$\alpha$' about $X$-axis i.e., $R_{(X,\alpha)}$

(ii) Rotation of angle '$\beta$' about $Z$-axis i.e., $R_{(Z,\beta)}$

(iii) Rotation of angle '$\gamma$' about $X$-axis i.e., $R_{(X,\gamma)}$

Where,

$$R_{(X,\alpha)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$R_{(Z,\beta)} = \begin{bmatrix} C\beta & -S\beta & 0 \\ S\beta & C\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{(X,\gamma)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}$$

Now, the resultant rotation matrix can be obtained by,

$$R = R_{(X,\alpha)}\, R_{(Z,\beta)}\, R_{(X,\gamma)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta & -S\beta & 0 \\ S\beta & C\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta(1)-S\beta(0)+0(0) & C\beta(0)-S\beta(C\gamma)+0(S\gamma) & C\beta(0)-S\beta(-S\gamma)+0(C\gamma) \\ S\beta(1)+C\beta(0)+0(0) & S\beta(0)+C\beta(C\gamma)+0(S\gamma) & S\beta(0)+C\beta(-S\gamma)+0(C\gamma) \\ 0(1)+0(0)+1(0) & 0(0)+0(C\gamma)+1(S\gamma) & 0(0)+0(-S\gamma)+1(C\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta & -S\beta C\gamma & S\beta S\gamma \\ S\beta & C\beta C\gamma & -C\beta S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}$$





$$= \begin{bmatrix} 1(C\beta)+0(S\beta)+0(0) & 1(-S\beta C\gamma)+0(C\beta C\gamma)+0(S\gamma) & 1(S\beta S\gamma)+0(-C\beta S\gamma)+0(C\gamma) \\ 0(C\beta)+C\alpha(S\beta)-S\alpha(0) & 0(-S\beta C\gamma)+C\alpha(C\beta C\gamma)-S\alpha(S\gamma) & 0(S\beta S\gamma)+C\alpha(-C\beta S\gamma)-S\alpha(C\gamma) \\ 0(C\beta)+S\alpha(S\beta)+C\alpha(0) & 0(-S\beta C\gamma)+S\alpha(C\beta C\gamma)+C\alpha(S\gamma) & 0(S\beta S\gamma)+S\alpha(-C\beta S\gamma)+C\alpha(C\gamma) \end{bmatrix}$$

$$R = \begin{bmatrix} C\beta & -S\beta C\gamma & S\beta S\gamma \\ C\alpha S\beta & C\alpha C\beta C\gamma - S\alpha S\gamma & -C\alpha C\beta S\gamma - S\alpha C\gamma \\ S\alpha S\beta & S\alpha C\beta C\gamma + C\alpha S\gamma & -S\alpha C\beta S\gamma + C\alpha C\gamma \end{bmatrix}$$

The above matrix is the resultant rotation of fixed axes $XZX$ corresponding to set of Euler's angle.

**Q23. Describe 3-degree of freedom of 2-D manipulator.**

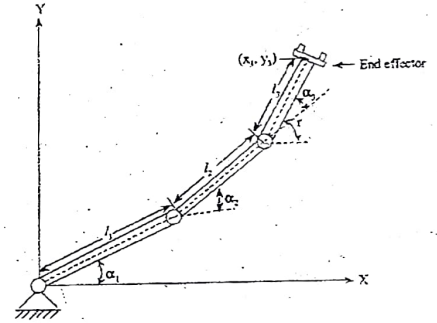
**Answer :**

In revolute joint, then end-effector position is specified by joint angles $\alpha_1$, $\alpha_2$, $\alpha_3$,.....

$P_{joint} = (\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\alpha_n)$

The end-effector position is specified on, $P_{space} = (x, y, z)$

The figure shows 3DOF of 2-D manipulator.




**Figure**

In the above figure,

$l_1, l_2$ and $l_3$ – Lengths of links

$\alpha_1, \alpha_2, \alpha_3$ – Joint angles

$r$ – Orientation.

**Forward Transformation**

The end effector's position and its orientation in space is determined from joint angles and length of links by the following equations,

$$x_3 = l_1 \cos\alpha_1 + l_2 \cos(\alpha_1 + \alpha_2) + l_3 \cos(\alpha_1 + \alpha_2 + \alpha_3)$$

$$y_3 = l_1 \sin\alpha_1 + l_2 \sin(\alpha_1 + \alpha_2) + l_3 \sin(\alpha_1 + \alpha_2 + \alpha_3)$$

$$r = \alpha_1 + \alpha_2 + \alpha_3$$

**Reverse Transformation**

By using reverse transformation, joint angles can be determined from the end effector's position $(x_3, y_3)$ and the orientation $(r)$ as follows.

$$x_2 = x_3 - l_3 \cos r \quad (1)$$

$$y_2 = y_3 - l_3 \sin r \quad (2)$$

From the geometry of the above figure,

$$x_2 = l_1 \cos\alpha_1 + l_2 \cos\alpha_1 . \cos\alpha_2 - l_2 \sin\alpha_1 . \sin\alpha_2 \quad (3)$$

$$y_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_1 . \cos\alpha_2 + l_2 \cos\alpha_1 . \sin\alpha_2 \quad (4)$$

By squaring and adding equations (3) and (4),

$$\cos\alpha_2 = \frac{x_2^2 + y_2^2 - l_1^2 - l_2^2}{2l_1l_2}$$

By substituting $\alpha_2$ value in equations (1) and (2), and on simplifying.

$\therefore \quad \alpha_3 = r - (\alpha_1 + \alpha_2)$

**Q24. Obtain the homogeneous transformation matrix that represents a rotation of 'α' degrees about the current X-axis followed by a translation of 'b' units along the current X-axis, followed by a translation 'd' units along the current Z-axis, followed by a rotation of 'θ' degrees about the current Z-axis?**

May/June-14, Set-2, Q3(a)

**Answer :**

Given that,

Rotating of 'α' degrees about $X$-axis,

$$\text{i.e., } R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation of '$b$' units along $X$-axis,

$$T_{x,b} = \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation of '$d$' units along $Z$-axis,

$$T_{z,d} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation of 'θ' degrees about Z-axis,

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The homogeneous transformation matrix is given as,

$$H = \{R_{x,\alpha}\}\{T_{x,b}\}\{T_{z,d}\}\{R_{z,\theta}\}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

On simplification,

$$H = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & b \\ \cos\alpha\sin\theta & \cos\alpha\cos\theta & -\sin\alpha & -\sin\alpha\, d \\ \sin\alpha\sin\theta & \sin\alpha\cos\theta & \cos\alpha & \cos\alpha\, d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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**Q25. If a point (8i + 5j + 6k) is translated 4 units along Y-axis and then rotated 30° about X-axis, obtain the coordinates after transformation.**

**Answer :** May/June-14, Set-3, Q3(b)

Given that,

Vector point, $P = 8i + 5j + 6k$

$\therefore$ The vector matrix is.

$$P = \begin{bmatrix} 8 \\ 5 \\ 6 \\ 1 \end{bmatrix}$$

The homogeneous transformation matrix for translation of 4 units along Y-axis and rotation of 30° about X-axis is given as,

$$T = \{R_{X,30^\circ}\}\{T_{Y,4}\}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ & 0 \\ 0 & \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 3.464 \\ 0 & 0.5 & 0.866 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed vector matrix after translation and rotation is given as,

$$P' = T.P$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.05 & 3.464 \\ 0 & 0.5 & 0.866 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 6 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 4.794 \\ 9.696 \\ 1 \end{bmatrix}$$

The co-ordiantes after transformation are,

$$P' = [8 \quad 4.794 \quad 9.696 \quad 1]^T$$

**Q26. For the point $P_{xyz} = (0, 5, 2)^T$ perform following operations (i) Rotate 30° about the Y-axis followed by translation of 4 units along z-axis (ii) Translate 6 units along Z-axis followed by rotation of 60° along y-axis.**

May/June-14, Set-4, Q3(b)

**Answer :**

Given that,

$$P_{xyz} = (0, 5, 2)^T$$

(i) Rotation about Y-axis for rotating 30° angle.

$$R_{Y,30^\circ} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 30^\circ & 0 & \sin 30^\circ \\ 0 & 1 & 0 \\ -\sin 30^\circ & 0 & \cos 30^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.866 \end{bmatrix}$$

Let,

$$a_1 = R_{Y30^\circ} \times P_{xyz}$$

$$= \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.866 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ 1.732 \end{bmatrix}$$

Translation about z-axis for a distance of 4 units.

$$d_z = [0, 0, 4]^T$$

Final position,

$$a_2 = a_1 + d_z$$

$$= \begin{bmatrix} 1 \\ 5 \\ 1.732 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ 5.732 \end{bmatrix}$$

(ii) Translation 6 units along z-axis followed by rotation 60° along y-axis

Translation about 6 units along z-axis,

$$d_y = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

But,

$$a_1 = P_{xyz} + d_z$$

$$= \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$$

Rotation about Y-axis around 60°.

$$R_{Y,60^\circ} = \begin{bmatrix} \cos 60^\circ & 0 & \sin 60^\circ \\ 0 & 1 & 0 \\ -\sin 60^\circ & 0 & \cos 60^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & 0.5 \end{bmatrix}$$

Final position,

$$a_2 = R_{Y,60^\circ} \times a_1$$

$$= \begin{bmatrix} 0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4.33 \\ 5 \\ 4 \end{bmatrix}$$

**Q27. Find the transformation matrices for the following operations on the point $2\hat{i} - 5\hat{j} - 3\hat{k}$.**

**(i) Rotate 60° about x-axis and then translate 5 units along y-axis**

**(ii) Translate –2 units along y-axis and rotate 45° about x-axis**

**(iii) Translate 6 units along x-axis and rotate 45° about z-axis.**

**Answer :** Nov.-10, Set-3, Q3(b)

The given vector is $u = 2i - 5j - 3k$.

$$\therefore \text{ Vector matrix, } u = \begin{bmatrix} 2 \\ -5 \\ -3 \\ 1 \end{bmatrix}$$

**(i) Rotate 60° About x-axis and Then Translate 5 Units Along y-axis**

The homogeneous transformation matrix ($T$) to perform rotation of 60° about x-axis and then translate 5 units along y-axis is given by,

$$T = \text{Tran}(y, 5)\ \text{Rot}(x, 60^\circ)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ & 0 \\ 0 & \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 5 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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The transformed vector after performing the above mentioned sequence of operations is given by,

$$V = Tu$$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.866 & 5 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5.098 \\ -5.83 \\ 1 \end{bmatrix}$$

**(ii) Translate –2 Units Along y-axis and Rotate 45° About x-axis**

The homogeneous transformation matrix ($T$) to perform translation of –2 units along y-axis and then rotation of 45° about x-axis,

$$T = \text{Rot}(x, 45^\circ)\ \text{Tran}(y, -2)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 45^\circ & -\sin 45^\circ & 0 \\ 0 & \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.707 & -0.707 & 0 \\ 0 & 0.707 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.707 & -0.707 & -1.414 \\ 0 & 0.707 & 0.707 & -1.414 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed vector after performing the above mentioned sequence of operations is given by,

$$V = Tu$$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.707 & -0.707 & -1.414 \\ 0 & 0.707 & 0.707 & -1.414 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2.828 \\ -7.070 \\ 1 \end{bmatrix}$$

**(iii) Translate 6 Units along x-axis and Rotate 45° about z-axis**

The homogeneous transformation matrix to perform translation of 6 units along x-axis and then rotation of 45° about z-axis is given by,

$$T = \text{Rot}(z, 45^\circ)\ \text{Tran}(x, 6)$$

$$T = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad T = \begin{bmatrix} 0.707 & -0.707 & 0 & 4.242 \\ 0.707 & 0.707 & 0 & 4.242 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed vector after performing the above mentioned sequence of operations is given by,

$$V = Tu$$

$$V = \begin{bmatrix} 0.707 & -0.707 & 0 & 4.242 \\ 0.707 & 0.707 & 0 & 4.242 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.191 \\ 2.121 \\ -3 \\ 1 \end{bmatrix}$$

**Q28. A point vector is translated in x, y and z directions by a distance of 1, 2 and 1 respectively. Write the transformation matrix for the translation. If the point is (3, 2, 1), determine the position of this point after translation. Afterwards the point vector is rotated about the x, y and z axes by angles of 30°, 45° and 60°, respectively. Write the transformation matrices for the rotations and also the total transformation matrix from the initial position of (3, 2, 1). Also find the new position of this point.**

April/May-13, Set-2. Q2

**Answer :**

Given that,

Translation = $[1, 2, 1]^T$

The transformation matrix for translation is given by,

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The position vector at point $P$ (3, 2, 1)

$$P_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1 \\ 2+2 \\ 1+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

Position vector, $P_{xyz} = [4, 4, 2]^T$

**Rotation Matrix**

The rotation matrix about $x$-axis is given by,

$$R(x, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(x, 30°) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° & 0 \\ 0 & \sin 30° & \cos 30° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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The rotation matrix about y-axis is given by,

$$Rot(y, 45°) = \begin{bmatrix} \cos 45° & 0 & \sin 45° & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 45° & 0 & \cos 45° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix about $z$-axis is given by,

$$Rot(z, 60°) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(60°) & -\sin 60° & 0 & 0 \\ \sin 60° & \cos 60° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(z, 60°) = \begin{bmatrix} 0.5 & -0.866 & 0 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrices for the rotations is given as,

$$T_1 = Rot(x, 30°).\ Rot(y, 45°).\ Rot(z, 60°)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.866 & 0 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 0.354 & -0.612 & 0.707 & 0 \\ -0.573 & 0.127 & -0.354 & 0 \\ -0.739 & 0.78 & 0.612 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The transformation matrix after initial position of (3, 2, 1).

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Total transformation,

$$T = T_1 \times T_2$$

$$= \begin{bmatrix} 0.354 & -0.612 & 0.707 & 0 \\ -0.573 & 0.127 & -0.354 & 0 \\ -0.739 & 0.78 & 0.612 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.354 & -0.612 & 0.707 & 0.545 \\ -0.573 & 0.127 & -0.354 & -1.819 \\ -0.739 & 0.78 & 0.612 & -0.045 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

New position of this point,

$$P = [0.545, -1.819, -0.045]^T$$

**Q29. Two frames A & B are initially coincident. Frame A is fixed, while frame B moves according to the following sequence.**

**(i) Rotation about $X_A$ of 30 degrees**

**(ii) Rotation about $Y_B$ of 60 degrees**

**(iii) Rotation about $Z_A$ of 90 degrees and**

**(iv) Translation of (1, 2, 3) along Frame B. Find the new position and orientation of frame B in frame A. (Express this as a homogeneous transformation matrix).**

**Answer :**

[April/May-13, Set-4, Q2 | Model Paper-III, Q7(a)]

**(i) Rotation about '$X_A$' of 30 degrees**

The rotation matrix is given as,

$$R_x(30) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 & 0 \\ 0 & \sin 30 & \cos 30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**(ii) Rotation about '$Y_B$' of 60 degrees**

$$R(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(60) = \begin{bmatrix} \cos 60 & 0 & \sin 60 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 60 & 0 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0.866 & 0 \\ 0 & 1 & 0 & 0 \\ -0.866 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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**(iii) Rotation about '$Z_A$' of 90 degrees**

$$R(Z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(90°) = \begin{bmatrix} \cos 90° & -\sin 90° & 0 & 0 \\ \cos 90° & \cos 90° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(90°) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**(iv) Translation of (1, 2, 3) along frame B**

$$R_{trans} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous transformation is given by,

$$T = R(30°).R(60°).R(90°).R_{trans}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0 & 0.866 & 0 \\ 0 & 1 & 0 & 0 \\ -0.866 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

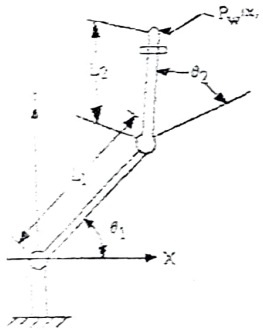
$$T = \begin{bmatrix} 0 & -0.5 & 0.866 & 1.598 \\ 0.866 & -0.433 & -0.25 & -0.75 \\ 0.5 & 0.75 & 0.433 & 3.299 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation of frame '$B$' with respect to $A$,

$$[n, s, a] = \begin{bmatrix} 0 & -0.5 & 0.866 & 1.598 \\ 0.866 & -0.433 & -0.25 & -0.75 \\ 0.5 & 0.75 & 0.433 & 3.299 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

New position vector = [1.598, –0.75, 3.299]

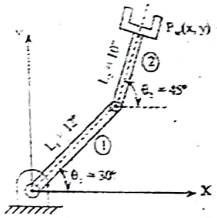
Q30. Consider the forward transformation of the two-joint manipulator shown in figure below. Given that the length of joint 1, $L_1$ = 12 in, the length of joint 2, $L_2$ = 10 in, the angle $\theta_1$ = 30° and the angle $\theta_2$ = 45°, compute the co-ordinate position (x and y co-ordinates) for the end-of-the-arm $P_w$.




Figure

**Answer :**

Forward transformation of the two-joint manipulator shown in figure.




Figure

Given the length of joint link, $L_1$ = 12 inches = 12"

Length of joint link, $L_2$ = 10 inches = 10"

Joint angles, $\theta_1 = 30°$

$\theta_2 = 45°$

The co-ordinates position for the end-of-the arm, $P_w(x, y)$ is as follows,

In revolute joint, the end-effector position is specified by joint angles $\theta_1, \theta_2, \theta_3 \ldots \theta_n$

$$P_{joint} = (\theta_1, \theta_2 \ldots \theta_n)$$

The required equations are,

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

The end effector orientation, $r = \theta_1 + \theta_2$

$$\therefore \quad x = 12 \times \cos 30° + 10 \times \cos(30° + 45°)$$

$$= 12 \times \frac{\sqrt{3}}{2} + 10 \times (0.258)$$

$$x = 6\sqrt{3} + 2.58$$

$$= 12.97 \text{ inches.}$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

$$= 12 \times \sin 30° + 10 \times \sin(30° + 45°)$$

$$= (12 \times 0.5) + (10 \times 0.966)$$

$$= 15.66 \text{ inches.}$$

The orientation of end of the arm $P_w$

$$r_z = \theta_1 + \theta_2$$

$$= 30° + 45°$$

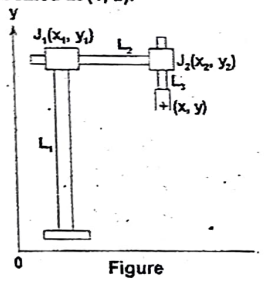
$$= 75°$$

Co-ordinate position $(x, y) = (12.98, 15.66)$

**Q31. A L-L robot has two links of variable length. Assuming that the global coordinate system is defined at joint $J_1$, determine:**

**(a) The coordinates of the end-effector point if the variable link lengths are 1 m and 2 m as shown in figure.**

**(b) Variable link lengths if the end-effector is located at (1, 2).**



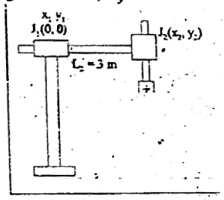

Figure

**Answer :** April/May-13, Set-3, Q2

Given that,

Length of link-2, $L_2$ = 1 m

Length of link-3, $L_3$ = 2 m




Figure





The transformation matrix is given by,

$$T_L = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & -L_3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

And, $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T_L \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (\because x_1 = 0, y_1 = 0)$$

$$\therefore \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(a) The coordinates of end-effector are,

$(x, y) = (1, -2)$

(b) The end effector is located at (1, 2)

$$L_2 = x - x_1$$

$$= 1 - 0$$

$$= 1 \text{ m}$$

And, $L_3 = -y + y_1$

$$= -2 + 0$$

$$= -2 \text{ m}$$

The minus sign indicates the coordinate system which is used.

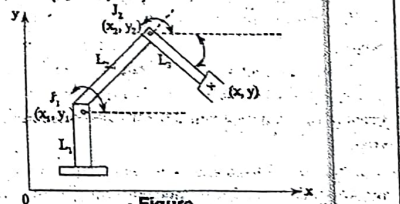
∴ Variable link lengths are,

$L_2$ = 1 m and $L_3$ = 2 m

**Q32. An RR robot has two links of length 0.50 m and 1.0 m, respectively. Assuming that the global coordinate system is defined at joint $J_1$, determine,**

**(a) The coordinates of the end-effector point if the joint rotations are 45° at both joints as shown in figure.**

**(b) Joint rotations if the end-effector is located at (1, 4. 4, 0.5)**




Figure

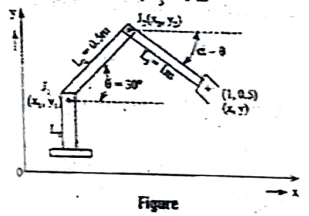
[April/May-13, Set-1, Q2 | Model Paper-V, Q5(b)]

**Answer :**

Given that,

Length of link-2, $L_2$ = 0.5 m

Length of link-3, $L_3$ = 1 m




Figure

(a) The transformation matrix is given by,

$$T_L = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & -L_3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_L = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T_L \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1 \end{bmatrix}$$

The coordinate of end effectors are $(x, y) = (1.5, -0.5)$.

(b) If the end-effector is located at (1, 0.5)

$$\cos\alpha = \frac{x^2 + y^2 - L_2^2 - L_3^2}{2L_2L_3}$$

$$\cos\alpha = \frac{1 + (0.5)^2 - (0.5)^2 - (1)^2}{2(0.5)(1)}$$

$$\cos\alpha = 0$$

$$\therefore \alpha = 90°$$

$$\tan\theta = \frac{(y - y_1)(L_2 + L_3\cos\alpha) + (x - x_1)L_3 \sin\alpha}{(x - x_1)(L_2 + L_3\cos\alpha) - (y - y_1)L_3 \sin\alpha}$$

$$= \frac{0.5(0.5 + 1.\cos 90°) + (1).1 \sin 90°}{(1)(0.5 + 1.\cos 90°) - (0.5).\sin 90°}$$

$$\frac{0.5(0.5) + 1}{0} = \infty$$

$$\therefore \tan\theta = \infty$$

$$\theta = 90°$$

Joint rotation, $\theta$ = 90°.

## 3.2 SPECIFICATIONS OF MATERIALS, D-H NOTATION, JOINT COORDINATES AND WORLD COORDINATES - FORWARD AND INVERSE KINEMATICS - PROBLEMS

### Q33. Explain about the world coordinate system.

**Answer :**

World Co-ordinate System (WCS) is used to describe the arm work space. In WCS, points in space are identified by Cartesian locations and motion parallel to any axis is available. The WCS is based on an axis system with three axes $X$, $Y$ and $Z$ at right angles to each other which intersects at the origin. The origin is the center of the robot mounting flange when a base offset is not set. To store world coordinate data, the software uses Cartesian locations. Coordinates contain data about position and orientation.

**Position**

The position of a point in the work space is identified by distances (positive or negative) along the $X$, $Y$ and $Z$-axes from the origin.

For example, (36.0, 0.0, 54.0)

The center of the mechanical interface is described as 36.0 cm in a positive $x$ direction, 0.0 in a $y$-direction and 54.0 cm in a positive $z$-direction.

**Orientation**

When the tool is at a point in space, it is orientated in different ways.

For example, it could be pointing downward (parallel to the negative $z$-axis) or pointing forward (parallel to the positive axis) etc.

The orientation of a tool is identified by a rotation (positive or negative) around the $X$, $Y$ and $Z$-axis. Rotation around the $X$-axis is called "roll" around the $Y$-axis is called "pitch" and around the $Z$-axis is called "yaw". By convection, these are written in order yaw, pitch, roll.

For example, the mechanical interface is oriented with no yaw, a 15.0 degree pitch and 40.0 degree roll

or (0.0, 15.0, –40.0).

**Full Co-ordinates**

The full coordinates for a tool point are written in the order,

$x, y, z$ yaw, pitch, roll.

Example, (36.0, 0.0, 54.0, 0.0, 15.0, – 40.0).

The first problem encountered is to describe both the position of the tool and the locations with respect to a common coordinate system. It is customery to establish a fixed coordinate system called the word or base frame to which all objects including the manipulator are referenced. It is to establish the base coordinate frame $o_o x_o y_o$ at the base of the robot, as shown in figure and the coordinates $(x, y)$ of the tool are expressed in coordinate frame as,

$$x = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$

$$y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

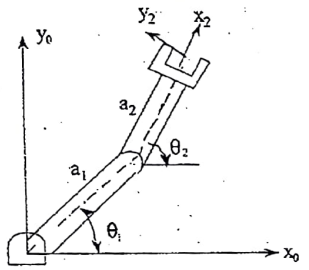



**Figure: Coordinate Frames for Two-link Planer Robot**

The orientation of the tool frame relative to the base frame is given by the direction cosines of the $x_2$ and $y_2$ axes relative to the $x_0$ and $y_0$ axes i.e.,

$$i_2 \cdot i_0 = \cos(\theta_1 + \theta_2)$$

$$i_2 \cdot j_0 = \sin(\theta_1 + \theta_2)$$

$$j_2 \cdot i_0 = -\sin(\theta_1 + \theta_2)$$

$$j_2 \cdot j_0 = \cos(\theta_1 + \theta_2)$$

$$\begin{bmatrix} i_2 \cdot i_0 & j_2 \cdot i_0 \\ i_2 \cdot j_0 & j_2 \cdot j_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Where, $i_0, j_0$ are the standard orthonormal unit vectors in base frame and $i_2, j_2$ are the standard orthonormal unit vector in tool frame.

### Q34. Write the steps involved in deriving forward kinematics for any manipulator based on D-H convention.

**Answer :** Model Paper-II, Q7(b)

**Denavit-Hartenberg Convention (D-H Convention)**

Denavit-Hartenberg convention procedure was introduced for an open kinematic manipulator chain with links and joints. In this manipulator each link is provided with a right handed orthogonal coordinate system. It is useful to obtain a homogeneous transformation matrix for each joint with respect to refernece coordinates system and helps in kinematic arm equation formulation.

**Step-by-Step Procedure of D-H Convention for a Manipulator**

**Step-1**

Determine the number of joints and links and give the number, starting from zero to '$l$'.

**Step-2**

Provide right handed coordinate system and make sure that $Z_n$ axis aligns with $(n+1)$ joint axis.

**Provide Coordinate Frames for the Intermediate Joints**

**Step-3**

$X_n$-axis is fixed and is at right angles to $Z_{n-1}$ and $Z_n$, away from $Z_{n-1}$. The intersection point of $Z_n$ and $x_n$ axes gives the position of origin for $\{n\}$. For this, three cases occur.



**Case (a)**

Mark the origin, if $Z_{n-1}$ and $Z_n$ axes intersect each other. $x_n$ axis will be at right angle to their plane, which results as, $a_n = 0$.

**Case (b)**

When $Z_{n-1}$ and $Z_n$ axes are either parallel to each other, or lie in parallel plane, their common normal is difficult to be determined. If '$n$' joint is revolute, then $x_n$ axis is made parallel to that normal and passes through origin of $\{n-1\}$. This results in fixed origin and makes $d_n = 0$. If '$n$' joint is prismatic, then $x_n$ axis is made any of the required normal and origin is determined at distal end of link '$n$'.

**Case (c)**

When $Z_{n-1}$ and $Z_n$ coincide, origin will be on common axis. If '$n$' is revolute, then its origin coincides with $\{n-1\}$, $x_n$ axis coincides with $x_{n-1}$ axis and makes $d_n = 0$. If '$n$' is prismatic, '$x_n$' axis is made parallel to $x_{n-1}$ aixs and makes $a_n = 0$ and origin is located at distal end of link '$n$'.

**Step-4**

$y_n$-axis is fixed in order to complete the coordinate system.

**Provide Frame to Link-0, i.e., Frame {0}**

**Step-5**

Frame {0} is located based on assumption and reference from workpiece. If $\theta_1 = 0$, then $x_0$-axis should be parallel to $x_1$-axis. If joint '1' is revolute, then origin can be positioned at an arbitrary reference to make $d_1 = 0$. If joint '1' is prismatic, $x_0$ and $x_1$ axes are parallel and $\theta_1 = 0$. Thus, the origin of frame {0} is placed arbitrarily based on joint '1'.

**Step-6**

$y_0$-axis completes the co-ordinate frame for frame {0}.

**Provide frame for end-effector i.e., frame {$l$}.**

**Step-7**

Origin is made at the end point (or) tool point of the manipulator and frame {$l$} is called tool frame.

**Step-8**

$Z_l$ is to be fixed parallel to $Z_{l-1}$ axis and away from link '$l$'. It is the direction of approach.

**Step-9**

If '$l$' is prismatic, then $x_l$ is made parallel to $x_{l-1}$ axis. If '$l$' is revolute, then $x_l$ is similar, to follow step-5, i.e., $x_l$ is right angled to both $Z_{l-1}$ and $Z_l$ axes. It in normal direction. Thus, $y_l$ axis is chosen to complete the coordinate system of frame {$l$}. In this, the $y_l$-axis indicates 'orientation' or 'sliding' direction.

### Q35. Derive the forward kinematics matrix for a robot arm without wrist.

**Answer :**

Forward kinematics is computation of the position and orientation of robot's end effector as a function of its joint angles.

For a serial chain of $n$ links, let $\theta_i$ be the angle of link $i$.

Consider, $\theta_0, \theta_1, \ldots, \theta_n$, the frame of link $n$ relative to link-0 is,

$$O_{T_n} = \prod_{i=1}^{n} {}^{i-1}T_i(\theta_i)$$

Where,

${}^{i-1}T_i(\theta_i)$ is the transformation matrix from the frame of link $i$ to link $i-1$.

In robotics, these are conventionally described by D-H (Denavit - Hartenberg) parameters.

Forward kinematics is used to describe the robot position and orientation of the manipulator linkages. Two ways to express position of any link.

(i) Using Cartesian space

(ii) Using joint space.

**(i) Cartesian Space**

It consists of position $(x, y, z)$ and orientation, represented by 3 × 3 matrix (rotation matrix).

**(ii) Joint Space**

It represents position by the angles of the manipulator links. Forward kinematics is the transformation from joint space to Cartesian space. This transformation depends on the configuration of the robot (i.e., link lengths, joint position, type of each joint etc.). To describe the location of each link relative to another link, a frame is attached to each link, then specify a set of parameters that characterize this frame, which is represented by D-H notation.

Forward kinematics uses to compute the position, orientation and velocity of the end effect or given the displacements and joint angles. In forward kinematics, given a configuration of the robot, expressed as coordinates in the parameterized configuration space and able to compute the locations of points on the robot in the physical world.

**Example**

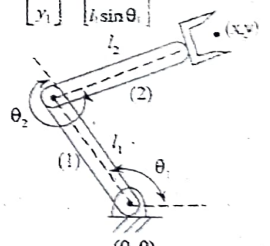
Consider a planer 2$R$ arm

The configuration is given by,

$(\theta_1, \theta_2) \in S \times S$

The location of the first joint is,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \end{bmatrix}$$




**Figure**

The location of the end effector is,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

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From notations, $\cos\theta_1 = c_1$

$\sin\theta_1 = s_1$

$\cos(\theta_1 + \theta_2) = c_{12}$

$\sin(\theta_1 + \theta_2) = s_{12}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1c_1 + l_2c_{12} \\ l_2s_1 + l_2s_{12} \end{bmatrix}$$

In general, the forward kinematics can be written in the form,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = f(q)$$

Where,

$f(q)$ is vector-valued function of a vector.

**Q36. Derive the forward kinematics equation using the D-H convention for the three link planar revolute jointed manipulator (RRR).**

May/June-14, Set-4, Q4

**Answer :**

For the three link planer (RRR) manipulator the D-H parameter table can be written as,

| Link | $\theta_i$ | $d_i$ | $a_i$ | $c_i$ |
|---|---|---|---|---|
| 1 | $\theta_1$ | 0 | $a_1$ | 0 |
| 2 | $\theta_2$ | 0 | $a_2$ | 0 |
| 3 | $\theta_3$ | 0 | $a_3$ | 0 |

The general transformation matrix,

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the given table.

For link 1,

$$^0T_1 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For link 2,

$$^1T_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For link 3,

$$^2T_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3C_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$^0T_2 = {}^0T_1\,{}^1T_2 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = \begin{bmatrix} C_1C_2 - S_1S_2 & -(C_1S_2 + S_1C_2) & 0 & a_2(C_1C_2 - S_1S_2) + a_1C_1 \\ S_1C_2 + C_1S_2 & C_1C_2 - S_1S_2 & 0 & a_2(S_1C_2 + C_1S_2) + a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = {}^0T_2\,{}^2T_3$$

$$= \begin{bmatrix} C_1C_2 - S_1S_2 & -(C_1S_2 + S_1C_2) & 0 & a_2(C_1C_2 - S_1S_2) + a_1C_1 \\ S_1C_2 + C_1S_2 & C_1C_2 - S_1S_2 & 0 & a_2(S_1C_2 + C_1S_2) + a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{12}C_3 - S_{12}S_3 & -(C_{12}S_3 + S_{12}C_3) & 0 & a_3(C_{12}C_3 - S_{12}S_3) + a_2C_{12} + a_1C_1 \\ S_{12}C_3 + C_{12}S_3 & C_{12}C_3 - S_{12}S_3 & 0 & a_3(S_{12}C_3 + C_{12}S_3) + a_2S_{12} + a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix can be written as,

$$\therefore\ ^0T_3 = \begin{bmatrix} C_{123} & -S_{123} & 0 & (a_3C_{123} + a_2C_{12} + a_1C_1) \\ S_{123} & C_{123} & 0 & (a_3S_{123} + a_2S_{12} + a_1S_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Q37. Explain about the inverse kinematics.**

**Answer :**

Inverse kinematics is a mathematical method to find the degrees of freedom of a system subjected to kinematic constraints. In manipulator robotics, inverse kinematics deals with the task of end effector position and orientation. The robot controller must solve a set of non-linear simultaneous algebraic equations.

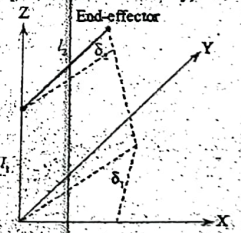
To solve inverse kinematics methods used are,

- Geometric
- Trigonometric
- Algebraic.

Inverse kinematics determine the possible lengths and angles from a predetermined goal position of the end-effector.

**Example**

Consider a family of robots ($l_1$, $l_2$ are the parameters of the family) with 2 degrees of freedom.




Figure

$l_1, l_2$ = Length of the two robot arms

$\rho_x, \rho_y, \rho_z$ = $x$, $y$ and $z$ coordinates of the position of the end-effector.

$\delta_1, \delta_2$ = Angles describing the rotations of the revolute joints.

$s_1, s_2, c_1, c_2$ = Sines and cosines of $\delta_1$, $\delta_2$ respectively.

Corresponding system of algebraic equations,

Given $l_1, l_2, \rho_x, \rho_z$

Solve for $s_1, c_1, s_2, c_2, \rho_y$

$$l_2 . c_1 . c_2 - \rho_x = 0$$
$$l_2 . s_1 . c_2 - \rho_y = 0$$
$$l_2 s_2 + l_1 - \rho_z = 0$$
$$c_1^2 + s_1^2 - 1 = 0$$
$$c_2^2 + s_2^2 - 1 = 0$$

According to Grobnes basis,

$$(l_2^2 - l_1^2 + 2l_1.\rho_z - \rho_z^2 - \rho_x^2) . c_1 - \rho_x . s_1 . \rho_y = 0$$
$$l_2^3 - l_2 l_1^2 + 2l_2 l_1 . \rho_2 - l_2 \rho_z^2 - l_2 \rho_x^2) . c_2 + (-l_2^2 + l_1^2 - 2l_1 \rho_z + \rho_z^2) . s_1 . \rho_y = 0$$
$$(\rho_2 - l_1^2 + 2l_1 \rho_z - \rho_z^2) . s_1^2 - l_2^2 + l_1^2 - 2l_1 \rho_z + \rho_z^2 + \rho_x^2 = 0.$$
$$l_2 S_2 + l_1 - \rho_z = 0$$
$$-l_2^2 + l_1^2 - 2l_1 \rho_z + \rho_z^2 + \rho_x^2 + \rho_y^2 = 0$$

So, for instance,

Consider,

$l_1$ = 30 cm (Length of first link)

$l_2$ = 45 cm (Length of second link)

$$\rho_x = \frac{45\sqrt{6}}{4}$$

$\simeq$ 27.5567 ($x$-coordinate of end-effector)

$$\rho_z = \frac{45\sqrt{2}}{2} + 30$$

$\simeq$ 61.8198 ($z$-coordinate of end-effector)

$$\rho_y = \frac{45\sqrt{2}}{4}, s_2 = \frac{\sqrt{2}}{2}$$

$$s_1 = \frac{1}{2}, c_2 = \frac{\sqrt{2}}{2}, c_1 = \frac{\sqrt{3}}{2}$$

i.e., the angles have to be set to $\delta_1 = 30°$, $\delta_2 = 45°$

Thus, inverse kinematics deals the known desired position of the tool, the joint values required to move the tool to the position.

**Q38. Derive D-H representation incase of a spherical wrist.**

**Answer :**

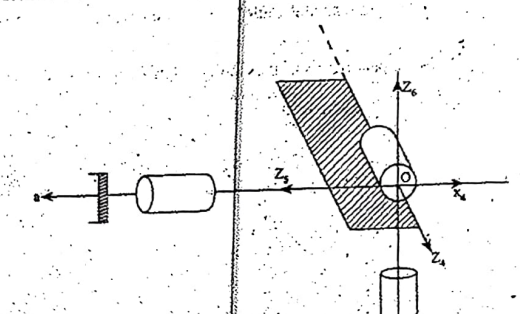



Figure: Spherical Wrist Arrangement

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The figure shows the spherical wrist coordiante frame. The origin '$O$' is formed by the intersection of joint axes $Z_4$ and $Z_5$.

From the figure, the final joint variables $\theta_4$, $\theta_5$, $\theta_6$ become Euler angles $\phi$, $\theta$, $\psi$ respectively by taking reference as coordinate frame $O_3x_3y_3z_3$. Thus, matrices $A_4$, $A_5$ and $A_6$ are given by,

$$A_4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

D-H parameters for spherical wrist are,

| Link | $a_n$ | $\alpha_n$ | $d_n$ | $\theta_n$ |
|---|---|---|---|---|
| 4 | 0 | –90 | 0 | $\theta_4$ |
| 5 | 0 | 90 | 0 | $\theta_5$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6$ |

(1) × (2) × (3),

$$T_3^6 = A_4 A_5 A_6 = \begin{bmatrix} R_3^6 & d_3^6 \\ 0 & 1 \end{bmatrix}$$

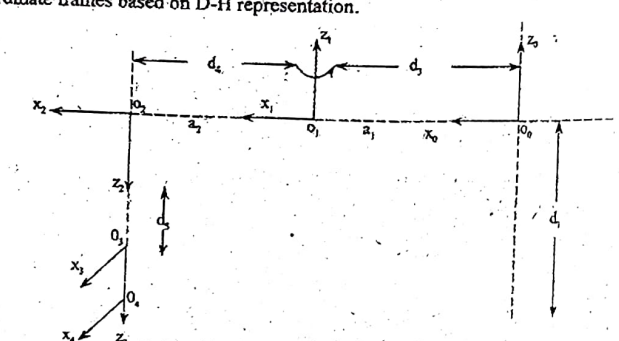
$$= \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 & C_4S_5d_6 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & S_4S_5d_6 \\ -S_5C_6 & S_5S_6 & C_5 & C_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the final equation of matrix, the joint variables $\theta_4$, $\theta_5$, $\theta_6$ are identified with the Euler angles $\phi$, $\theta$, $\psi$ by taking reference frame as $O_3x_3y_3z_3$.

**Q39. Draw and explain the link co-ordinate diagram of four axis SCARA robot.**

**Answer :**

(a) Assign of co-ordinate frames based on D-H representation.




Figure

(b) The parameter's table is,

| Joint | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | – |
| 2 | $a_2$ | 180 | 0 | – |
| 3 | 0 | 0 | – | 0 |
| 4 | 0 | 0 | $d_5$ | – |

(c) All the $A_i$ matrices are,

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) The ${}^0T_H$ is given by,

$${}^0T_H = A_1 \ldots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Q40. Show and explain the D-H parameters for $i^{th}$ link in a manipulator. Obtain the transformation matrix from frame i to frame i –1 attached to the ends of $i^{th}$ link in a manipulator in terms of the D-H parameters.** Nov.-10, Set-3, Q4(a)

**Answer :**

**Transformation Matrix**

Consider $\{i\}$ and $\{i-1\}$ be the frames associated with links $[i]$ and $[i-1]$ attached to the ends of $i^{th}$ link in a manipulator, shown in figure.

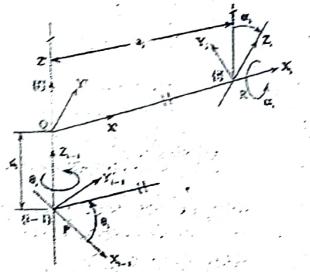


**Figure: Geometric Relationship between the Links**





D-H parameter with respect to frame $\{i\}$ and frame $\{i-1\}$ are $\theta_i$, $d_i$, $\alpha_i$, and $a_i$.

Where,

$\theta_i$ = Joint angle

$d_i$ = Joint distance

$\alpha_i$ = Link twist angle

$a_i$ = Link length.

Transformation of frame $\{i\}$ to frame $\{i-1\}$ involves four basic transformations. They are,

(a) Rotation of angle '$\theta_i$' about $z_{i-1}$ axis

(b) Translation along $z_{i-1}$ axis by distance '$d_i$'

(c) Translation along $x_i$ axis by distance '$a_i$'

(d) Rotation of angle '$\alpha_i$' about $x_i$ axis.

Transformation matrix with respect to frame $\{i\}$ to frame $\{i-1\}$ can be obtained by,

$${}^{i-1}T_i = T_{(z,\theta_i)}\, T_{(z,d_i)}\, T_{(x,a_i)}\, T_{(x,\alpha_i)}$$

Where,

$$T_{(z,\theta_i)} = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{(z,d_i)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{(z,a_i)} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{(x,\alpha_i)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(1)+0(0)+0(0)+a_i(0) & 1(0)+0(C\alpha_i)+0(S\alpha_i)+a_i(0) \\ 0(1)+1(0)+0(0)+0(0) & 0(0)+1(C\alpha_i)+0(S\alpha_i)+0(0) \\ 0(1)+0(0)+1(0)+0(0) & 0(0)+0(C\alpha_i)+1(S\alpha_i)+0(0) \\ 0(1)+0(0)+0(0)+1(0) & 0(0)+0(C\alpha_i)+0(S\alpha_i)+1(0) \end{bmatrix}$$

$$\begin{bmatrix} 1(0)+0(-S\alpha_i)+0(C\alpha_i)+a_i(0) & 1(0)+0(0)+0(0)+a_i(1) \\ 0(0)+1(-S\alpha_i)+0(C\alpha_i)+0(0) & 0(0)+1(0)+0(0)+0(1) \\ 0(0)+0(-S\alpha_i)+1(C\alpha_i)+0(0) & 0(0)+0(0)+1(0)+0(1) \\ 0(0)+0(-S\alpha_i)+0(C\alpha_i)+1(0) & 0(0)+0(0)+0(0)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(1)+0(0)+0(0)+0(0) & 1(0)+0(C\alpha_i)+0(S\alpha_i)+0(0) \\ 0(1)+1(0)+0(0)+0(0) & 0(0)+1(C\alpha_i)+0(S\alpha_i)+0(0) \\ 0(1)+0(0)+1(0)+d_i(0) & 0(0)+0(C\alpha_i)+1(S\alpha_i)+d_i(0) \\ 0(1)+0(0)+0(0)+1(0) & 0(0)+0(C\alpha_i)+0(S\alpha_i)+1(0) \end{bmatrix}$$

$$\begin{bmatrix} 1(0)+0(-S\alpha_i)+0(C\alpha_i)+0(0) & 1(a_i)+0(0)+0(0)+0(1) \\ 0(0)+1(-S\alpha_i)+0(C\alpha_i)+0(0) & 0(a_i)+1(0)+0(0)+0(1) \\ 0(0)+0(-S\alpha_i)+1(C\alpha_i)+d_i(0) & 0(a_i)+0(0)+1(0)+d_i(1) \\ 0(0)+0(-S\alpha_i)+0(C\alpha_i)+1(0) & 0(a_i)+0(0)+0(0)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_i(0)-S\theta_i(0)+0(0)+0(0) & C\theta_i(0)-S\theta_i(C\alpha_i)+0(S\alpha_i)+0(0) \\ S\theta_i(0)+C\theta_i(0)+0(0)+0(0) & S\theta_i(0)+C\theta_i(C\alpha_i)+0(S\alpha_i)+0(0) \\ 0(1)+0(0)+1(0)+0(0) & 0(0)+0(C\alpha_i)+1(S\alpha_i)+0(0) \\ 0(1)+0(0)+0(0)+1(0) & 0(0)+0(C\alpha_i)+0(S\alpha_i)+1(0) \end{bmatrix}$$

$$\begin{bmatrix} C\theta_i(0)-S\theta_i(-S\alpha_i)+0(C\alpha_i)+0(0) & C\theta_i(a_i)-S\theta_i(0)+0(d_i)+0(1) \\ S\theta_i(0)+C\theta_i(-S\alpha_i)+0(C\alpha_i)+0(0) & S\theta_i(a_i)+C\theta_i(0)+0(d_i)+0(1) \\ 0(0)+0(-S\alpha_i)+1(C\alpha_i)+0(0) & 0(a_i)+0(0)+1(d_i)+0(1) \\ 0(0)+0(-S\alpha_i)+0(C\alpha_i)+1(0) & 0(a_i)+0(0)+0(d_i)+1(1) \end{bmatrix}$$

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

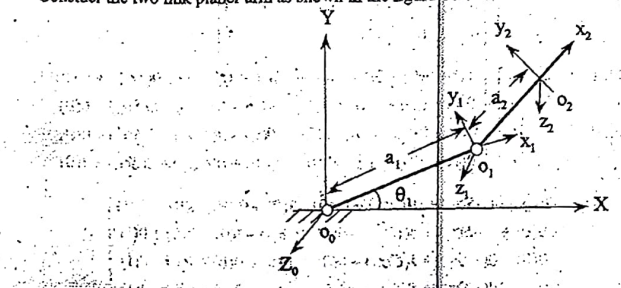
**Q41. Represent D-H parameters on a link of robot manipulator and determine them for a two-link planer arm.**

Nov.-11, Set-1, Q4(a)

**Answer :**

**Direct Kinematic Equations for the Two Link Planer Arm**

Consider the two link planer arm as shown in the figure below.




Figure



The joint axes $z_0$ and $z_1$ are normal. The final frame $o_2, x_2, y_2, z_2$ is fixed by choosing the origin $o_2$ at the end of link-2 as shown.

The D-H table for the two link planer arm is given below:

| Link | Link Length | Twist Angle | Joint Distance | Joint Angle |
|---|---|---|---|---|
| 'i' | '$a_i$' | '$\alpha_i$' | $d_i$ | $\theta_i$ |
| 1 | $a_1$ | 0 | 0 | $\theta_1$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2$ |

The two homogeneous transformation matrices are given by,

$$^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The solution to the forward kinematics position analysis is given by,

$$^0_2T = {}^0_1T\,{}^1_2T$$

$$= \begin{bmatrix} C_1C_2-S_1S_2+0+0 & -C_1S_2+(-S_1)C_2+0+0 & 0 & C_1a_2C_2-S_1a_2S_2+a_1C_1 \\ S_1C_2+C_1S_2+0+0 & -S_1S_2+C_1C_2+0+0 & 0 & S_1(a_2C_2)+C_1(a_2S_2)+a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{12} & -S_{12} & 0 & a_1C_1+a_2C_{12} \\ S_{12} & C_{12} & 0 & a_1S_1+a_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let, the known position and orientation of the end point of arm be,

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where, each $r_{ij}$ and $t_i$ has a numeric value.

The solution can be obtained by equating the above matrix with previously found homogeneous transformation matrix i.e.,

$$\begin{bmatrix} C_{12} & -S_{12} & 0 & a_1C_1+a_2C_{12} \\ S_{12} & C_{12} & 0 & a_1S_1+a_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Comparing the above two matrices,

$$t_1 = a_1C_1 + a_2C_{12}$$

$$t_2 = a_1S_1 + a_2S_{12}$$

$$t_3 = 0$$



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On squaring and adding the values of $t_1$ and $t_2$,

$$t_1^2 + t_2^2 = (a_1C_1 + a_2C_{12})^2 + (a_1S_1 + a_2S_{12})^2$$

$$t_1^2 + t_2^2 = a_1^2\cos^2\theta_1 + a_2^2\cos^2(\theta_1 + \theta_2) + 2a_1a_2\cos\theta_1\cos(\theta_1 + \theta_2) + a_1^2\sin^2\theta_1 + a_2^2\sin^2(\theta_1 + \theta_2) + 2a_1a_2\sin\theta_1\sin(\theta_1 + \theta_2)$$

$$t_1^2 + t_2^2 = a_1^2(\cos^2\theta_1 + \sin^2\theta_1) + a_2^2(\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)) + 2a_1a_2\cos\theta_2$$

$$t_1^2 + t_2^2 = a_1^2 + a_2^2 + 2a_1a_2C_2$$

$$C_2 = \left(\frac{t_1^2 + t_2^2 - a_1^2 - a_2^2}{2a_1a_2}\right) \text{ and}$$

$$S_2 = \pm\sqrt{1 - C_2^2}$$

Dividing on both sides with $C_2$.

$$\frac{S_2}{C_2} = \frac{\pm\sqrt{1 - C_2^2}}{C_2}$$

$$\boxed{\theta_2 = A\tan 2(S_2, C_2)}$$ [($A$ tan2) function is arc tangent]

For getting the value of first angle θ,

$$t_1 = a_1C_1 + a_2C_{12}$$

$$= a_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + a_1\cos\theta_1$$

$$= a_2(C_1C_2 - S_1S_2) + a_1C_1$$

$$t_1 = (a_1 + a_2C_2)C_1 - a_2S_1S_2$$

Similarly,

$$t_2 = a_1S_1 + a_2S_{12}$$

$$= a_2(S_1C_2 + C_1S_2) + a_1S_1$$

$$t_2 = (a_1 + a_2C_2)S_1 + a_2S_2C_1$$

Dividing $t_2$ by $t_1$,

$$\frac{t_2}{t_1} = \frac{(a_1 + a_2C_2)S_1 + a_2S_2C_1}{(a_1 + a_2C_2)C_1 - a_2S_1S_2}$$

Dividing numerator and denominator by $(a_1 + a_2C_2)C_1$,

$$\frac{t_2}{t_1} = \frac{\dfrac{(a_1 + a_2C_2)S_1}{(a_1 + a_2C_2)C_1} + \dfrac{a_2S_2C_1}{(a_1 + a_2C_2)C_1}}{\dfrac{(a_1 + a_2C_2)C_1}{(a_1 + a_2C_2)C_1} - \dfrac{a_2S_1S_2}{(a_1 + a_2C_2)C_1}}$$

$$\frac{t_2}{t_1} = \frac{\dfrac{S_1}{C_1} + \dfrac{a_2S_2}{a_1 + a_2C_2}}{1 - \dfrac{a_2S_2}{a_1 + a_2C_2}\cdot\dfrac{S_1}{C_1}}$$

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$$\frac{t_2}{t_1} = \frac{\tan\theta_1 + \dfrac{a_2S_2}{a_1 + a_2C_2}}{1 - \tan\theta_1\dfrac{a_2S_2}{a_1 + a_2C_2}}$$

Let, $\dfrac{a_2S_2}{a_1 + a_2C_2} = \tan\psi$

$$\frac{t_2}{t_1} = \frac{\tan\theta_1 + \tan\psi}{1 - \tan\theta_1\tan\psi} = \tan(\theta_1 + \psi) \quad \left[\because \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}\right]$$

$$\theta_1 = \tan^{-1}\left(\frac{t_2}{t_1}\right) - \psi$$

$$\boxed{\theta_1 = \tan^{-1}\left(\frac{t_2}{t_1}\right) - \tan^{-1}\left(\frac{a_2S_2}{a_1 + a_2C_2}\right)}$$

Therefore, arm solution is,

$$\theta_1 = \tan^{-1}\left(\frac{t_2}{t_1}\right) - \tan^{-1}\left(\frac{a_2S_2}{a_1 + a_2C_2}\right)$$

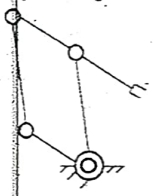
$$\theta_2 = A\tan 2(S_2, C_2)$$

Where,

$$C_2 = \left(\frac{t_1^2 + t_2^2 - a_1^2 - a_2^2}{2a_1a_2}\right) \text{ and}$$

$$S_2 = \pm\sqrt{1 - C_2^2}$$

**Q42. Find the direct kinematic equations for the parallelogram arm shown in figure,**

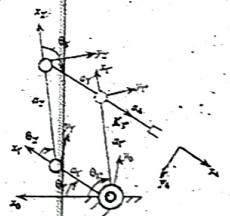


Figure

Nov-12, Set-1, 04

**Answer :**

Consider the parallelogram arm as shown in the figure. A closed chain is formed when link 1' and link1" is connected to link, respectively. Joint 4 is selected as the cut joint and the link frames have formed accordingly.



The D-H parameters for the parallelogram arm is in the table below.

| Link | Link length | Link Twist | Joint Distance | Joint Angle |
|---|---|---|---|---|
| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
| 1' | $a_{1'}$ | 0 | 0 | $\theta_{1'}$ |
| 2' | $a_{2'}$ | 0 | 0 | $\theta_{2'}$ |
| 3' | $a_{3'}$ | 0 | 0 | $\theta_{3'}$ |
| 1'' | $a_{1''}$ | 0 | 0 | $\theta_{1''}$ |
| 4 | $a_4$ | 0 | 0 | 0 |

Table

For parallelogram arm,

$a_{1'} = a_{3'}$ and $a_{2'} = a_{1''}$ and parameters of link 4 are all constant.

The homogeneous transformation matrix is defined by,

$$A_i^{i-1}(\theta_i) = \begin{bmatrix} C_i & -S_i & 0 & a_i C_i \\ S_i & C_i & 0 & a_i S_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where $C_i = \cos\theta_i$ and $S_i = \sin\theta_i$

For $A_1^0$, substitute $i = 1'$ in above equation

For $A_{2'}^{1'}$, substitute $i = 2'$ in above equation

For $A_{3'}^{2'}$, substitute $i = 3'$ in above equation

Therefore, the coordinate transformation for the two branches of the tree are shown respectively,

$$A_{3'}^0(q') = A_{1'}^0 A_{2'}^{1'} A_{3'}^{2'}$$

$$A_{3'}^0(q') = \begin{bmatrix} C_{1'2'3'} & -S_{1'2'3'} & 0 & a_{1'}C_{1'} + a_{2'}C_{1'2'} + a_{3'}C_{1'2'3'} \\ S_{1'2'3'} & C_{1'2'3'} & 0 & a_{1'}S_{1'} + a_{2'}S_{1'2'} + a_{3'}S_{1'2'3'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where, $q' = [\theta_{1'}\theta_{2'}\theta_{3'}]^T$ and $C_{1'2'3'} = \cos(\theta_{1'} + \theta_{2'} + \theta_{3'})$

$$A_{1''}^0(q'') = \begin{bmatrix} C_{1''} & -S_{1''} & 0 & a_{1''}C_{1''} \\ S_{1''} & C_{1''} & 0 & a_{1''}S_{1''} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where, $q'' = \theta_{1''}$

The constant homogeneous transformation for the last link is,

$$A_4^{3'} = \begin{bmatrix} 1 & 0 & 0 & a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \because C_4 = \cos\theta_4 = 1 \\ S_4 = \sin\theta_4 = 0 \end{bmatrix}$$

The position constraints are $(d_{3'1''} = 0)$

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i.e., if joint '$j + 1$' is revolute the constraints are,

$$R_i^j(q')\left(p_j^i(q') - p_k^i(q'')\right) = [0\ 0\ d_{jk}]^T$$

$$z_j^i(q') = z_k^i(q'')$$

$$R_0^{3'}(q')\left(p_{3'}^0(q') - p_{1''}^0(q'')\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The orientation constraints are satisfied independently of $q'$ and $q''$. As $a_{1'} = a_{3'}$ and $a_{2'} = a_{1''}$, the following other constraints are formed,

$$a_{1'}(C_{1'} + C_{1'2'3'}) + a_{1''}(C_{1'2'} - C_{1''}) = 0$$

$$a_{1'}(S_{1'} + S_{1'2'3'}) + a_{1''}(S_{1'2'} - S_{1''}) = 0$$

To satisfy these conditions for any value of $a_{1'}$ and $a_{1''}$, it must be,

$$\theta_{2'} = \theta_{1''} - \theta_{1'}$$

$$\theta_{3'} = \pi - \theta_{2'} = \pi - \theta_{1''} + \theta_{1'}$$

$$\theta_{3'} = \pi - \theta_{1''} + \theta_{1'}$$

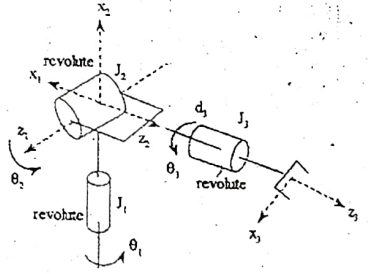
Therefore, the vector of joint variables is $q = [\theta_{1'}\theta_{1''}]^T$

Substituting the values of $\theta_{2'}$ and $\theta_{3'}$ in the homogeneous transformation $A_3^0$. We can find the direct kinematics function as,

$$T_4^0(q) = A_{3'}^0(q)A_4^{3'} = \begin{bmatrix} -C_{1'} & S_{1'} & 0 & a_{1''}C_{1''} - a_4C_{1'} \\ -S_{1'} & -C_{1'} & 0 & a_{1''}S_{1''} - a_4S_{1'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This shows that the parallelogram arm is kinematically equivalent to a two-link plannar arm.

**Q43. Obtain the D-H link parameters for the three-dimensional three degrees of freedom manipulator shown in figure. Find the homogeneous transformation matrix that describes the position and orientation of the end of arm with respect to the base frame.**




Figure

**Answer :** [Nov.-10, Set-4, Q4 | Model Paper-I, Q4]

For three-dimensional three degrees of freedom manipulator, D-H parameters are tabulated in the table

| Link | $\theta_i$ | $d_i$ | $a_i$ | $\alpha_i$ |
|---|---|---|---|---|
| 1 | $\theta_1$ | 0 | 0 | –90 |
| 2 | $\theta_2$ | 0 | 0 | 90 |
| 3 | $\theta_3$ | $d_3$ | 0 | 0 |

Table: D-H Parameters

The general transformation matrix of manipulator is given by,

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the table,

Transformation matrix for link-1 is obtained as,

$$^{0}T_1 = \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For Link-2

$$^{1}T_2 = \begin{bmatrix} C\theta_2 & 0 & S\theta_2 & 0 \\ S\theta_2 & 0 & -C\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For Link-3

$$^{2}T_3 = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, transformation matrix for the arm is,

$$^{0}T_3 = {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3$$

$$= \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & 0 & S\theta_2 & 0 \\ S\theta_2 & 0 & -C\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C\theta_1(C\theta_2)+0(S\theta_2)+S\theta_1(0)+0(0) & C\theta_1(-S\theta_2)+0(C\theta_2)+S\theta_1(0)+0(0) & C\theta_1(0)+0(0)+S\theta_1(1)+0(0) & C\theta_1(0)+0(0)+S\theta_1(d_3)+0(1) \\ S\theta_1(C\theta_2)+0(S\theta_2)-C\theta_1(0)+0(0) & S\theta_1(-S\theta_2)+0(C\theta_2)-C\theta_1(0)+0(0) & S\theta_1(0)+0(0)-C\theta_1(1)+0(0) & S\theta_1(0)+0(0)-C\theta_1(d_3)+0(1) \\ 0(C\theta_2)+1(S\theta_2)+0(0)+0(0) & 0(-S\theta_2)+1(C\theta_2)+0(0)+0(0) & 0(0)+1(0)+0(1)+0(0) & 0(0)+1(0)+0(d_3)+0(1) \\ 0(C\theta_2)+0(S\theta_2)+0(0)+1(0) & 0(-S\theta_2)+0(C\theta_2)+0(0)+1(0) & 0(0)+0(0)+0(1)+1(0) & 0(0)+0(0)+0(d_3)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2C\theta_3 & -C\theta_2S\theta_3 & S\theta_2 & d_3S\theta_2 \\ S\theta_2C\theta_3 & -S\theta_2S\theta_3 & -C\theta_2 & -d_3C\theta_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





$$= \begin{bmatrix} C\theta_1(C\theta_2C\theta_3)+0(S\theta_2C\theta_3)-S\theta_1(S\theta_3)+0(0) & C\theta_1(-C\theta_2S\theta_3)+0(-S\theta_2S\theta_3)-S\theta_1(C\theta_3)+0(0) \\ S\theta_1(C\theta_2C\theta_3)+0(S\theta_2C\theta_3)+C\theta_1(S\theta_3)+0(0) & S\theta_1(-C\theta_2S\theta_3)+0(-S\theta_2S\theta_3)+C\theta_1(C\theta_3)+0(0) \\ 0(C\theta_2C\theta_3)-1(S\theta_2C\theta_3)+0(S\theta_3)+0(0) & 0(-C\theta_2S\theta_3)-1(-S\theta_2S\theta_3)+0(C\theta_3)+0(0) \\ 0(C\theta_2C\theta_3)+0(S\theta_2C\theta_3)+0(S\theta_3)+1(0) & 0(-C\theta_2S\theta_3)+0(-S\theta_2S\theta_3)+0(C\theta_3)+1(0) \end{bmatrix}$$

$$\begin{bmatrix} C\theta_1(S\theta_2)+0(-C\theta_2)-S\theta_1(0)+0(0) & C\theta_1(d_3S\theta_2)+0(-d_3C\theta_2)-S\theta_1(0)+0(1) \\ S\theta_1(S\theta_2)+0(-C\theta_2)+C\theta_1(0)+0(0) & S\theta_1(d_3S\theta_2)+0(-d_3C\theta_2)+C\theta_1(0)+0(1) \\ 0(S\theta_2)-1(-C\theta_2)+0(0)+0(0) & 0(d_3S\theta_2)-1(-d_3C\theta_2)+0(0)+0(1) \\ 0(S\theta_2)+0(-C\theta_2)+0(0)+1(0) & 0(d_3S\theta_2)+0(-d_3C\theta_2)+0(0)+1(1) \end{bmatrix}$$

$$^{0}T_3 = \begin{bmatrix} C\theta_1C\theta_2C\theta_3 - S\theta_1S\theta_3 & -C\theta_1C\theta_2S\theta_3 - S\theta_1C\theta_3 & C\theta_1S\theta_2 & d_3C\theta_1S\theta_2 \\ S\theta_1C\theta_2C\theta_3 + C\theta_1S\theta_3 & -S\theta_1C\theta_2S\theta_3 + C\theta_1C\theta_3 & S\theta_1S\theta_2 & d_3S\theta_1S\theta_2 \\ -S\theta_2C\theta_3 & S\theta_2S\theta_3 & C\theta_2 & d_3C\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$C\theta_1 = \cos\theta_1$, $S\theta_1 = \sin\theta_1$

$C\theta_2 = \cos\theta_2$, $S\theta_2 = \sin\theta_2$

$C\theta_3 = \cos\theta_3$, $S\theta_3 = \sin\theta_3$.

**Q44. Derive the forward kinematics matrix for an articulated robot arm (3-axis) using D-H convention.**

**Answer :** May/June-14, Set-3, Q4

The forward kinematics referred to a method where the point variables (or) angles of the manipulator (i.e., $\theta_1$, $\theta_2$, $\theta_3$, ....) are known. This method is employed to determine the position and orientation for the end effector with reference to the base frame. Mathematically, the orientation and position can be obtained by the homogeneous transformation matrices of each link.

i.e., $T = \left[\begin{array}{c|c} R_{3\times3} & D_{3\times1} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right]$

Where,

$R_{3\times3}$ = Rotation matrix of 3 × 3 order (for orientation)

$D_{3\times1}$ = Vector matrix of 3 × 1 order (for position).

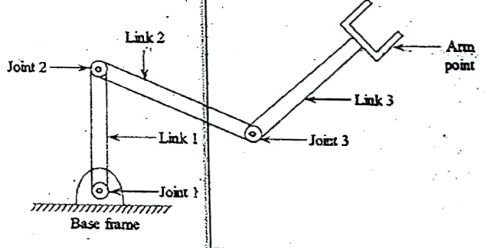



Figure (1)

Let $a_1$, $a_2$, $a_3$ are the length of respective links. Therefore, to calculate the position and orientation of arm point, consider the frame as shown in figure (2).

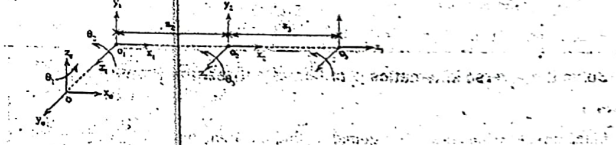


Figure (2)

From the above figure, joint link parameters for articulated arm can be formulated as,

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 90° | 0 | $\theta_1$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2$ |
| 3 | $a_3$ | 0 | 0 | $\theta_3$ |

Transformation matrix can be written as,

$$i-1_{U_i} = \begin{bmatrix} C\theta_i & -S\theta_i \cdot C\alpha_i & S\theta_i S\alpha_i & \alpha_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for link-1,

$$^0U_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for link-2,

$$^1U_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

The transformation matrix for link-3,

$$^2U_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3 C_3 \\ S_3 & C_3 & 0 & a_3 S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation matrix for the end point of articulated arm is given as,

$$^0U_3 = {}^0U_1 \times {}^1U_2 \times {}^2U_3$$

On multiplying,

$$^0U_3 = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & C_1(a_3 C_{23} + a_2 C_2) \\ S_1 C_{23} & -S_1 S_{23} & -C_1 & S_1(a_3 C_{23} + a_2 C_2) \\ S_{23} & C_{23} & 0 & a_3 S_{23} + a_1 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where,

$C_{23} = \cos(\theta_2 + \theta_3)$

$S_{23} = \sin(\theta_2 + \theta_3)$

At home position, i.e., $\theta_1 = \theta_2 = \theta_3 = 0$

**Q45. Solve the inverse kinematics problem of a three link Planer arm.**

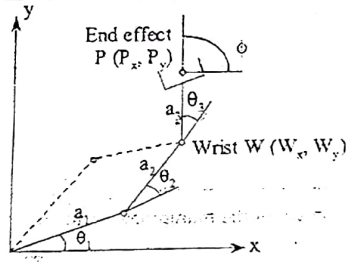
[Nov.-11, Set-1, Q4(b) | Model Paper-III, Q6(a)]

**Answer :**

In the inverse kinematics, it is required to find the joint angles $Q_1, Q_2, Q_3$ when the position and inclination of the end effector are given,








Figure

Consider a three Planer arm as shown in the above figure. Let the coordinates of the end effector and wrist be $P(P_x, P_y)$ and $W(W_x, W_y)$ respectively. Let $\phi$ be the inclination of the end effector. Let $a_1, a_2, a_3$ be the length of three links respectively

From the figure, $\phi = \theta_1 + \theta_2 + \theta_3$ ...(1)

And $P_x = a_1 C_1 + a_2 C_{12} + a_3 C_{123}$ ...(2)

$P_y = a_1 S_1 + a_2 S_{12} + a_3 S_{123}$ ...(3)

Considering the coordinates of wrist $W(W_x, W_y)$,

$W_x = P_x - a_3 C\phi = a_1 C_1 + a_2 C_{12}$ ...(4)

$W_y = P_y - a_3 S\phi = a_1 S_1 + a_2 S_{12}$ ...(5)

Squaring equations (4), (5) and adding them,

$$W_x^2 + W_y^2 = (a_1 C_1 + a_2 C_{12})^2 + (a_1 S_1 + a_2 S_{12})^2$$

$$W_x^2 + W_y^2 = a_1^2 C_1^2 + a_2^2 C_{12}^2 + 2a_1 a_2 C_1 C_{12} + a_1^2 S_1^2 + a_2^2 S_{12}^2 + 2a_1 a_2 S_1 S_{12}$$

$$W_x^2 + W_y^2 = a_1^2 (C_1^2 + S_1^2) + a_2^2 (C_{12}^2 + S_{12}^2) + 2a_1 a_2 (C_1 C_{12} + S_1 S_{12})$$

$$W_x^2 + W_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 C_2$$

$$C_2 = \frac{W_x^2 + W_y^2 - a_1^2 - a_2^2}{2a_1 a_2} \quad \text{...(6)}$$

For the point $P(P_x, P_y)$ to be within the workspare of the arm, we have to impose the condition that $-1 \le C_2 \le 1$.

And also $S_2 = \pm\sqrt{1 - C_2^2}$ ...(7)

Therefore, the angle $\boxed{\theta_2 = \text{Tan}^{-1}\left(\frac{S_2}{C_2}\right)}$ ...(8)

Expanding equations (4) and (5),

$W_x = a_1 C_1 + a_2 (C_1 C_2 - S_1 S_2)$

$W_x = (a_1 + a_2 C_2) C_1 - a_2 S_1 S_2$ ...(9)

And $W_y = a_1 S_1 + a_2 (S_1 C_2 + C_1 S_2)$

$W_y = (a_1 + a_2 C_2) S_1 + a_2 C_1 S_2$ ...(10)

Multiplying equation (9) with $a_2 S_2$ and equation (10) with $(a_1 + a_2 C_2)$ and subtracting them,

$$S_1 = \frac{(a_1 + a_2 C_2) W_y - a_2 S_2 W_x}{\Delta} \quad \text{...(11)}$$

Similarly,

$$C_1 = \frac{(a_1 + a_2 C_2) W_x - a_2 S_2 W_y}{\Delta} \quad \text{...(12)}$$

Where, $\Delta = a_1^2 + a_2^2 + 2a_1 a_2 C_2 = W_x^2 + W_y^2$

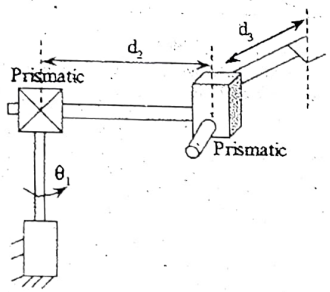
Dividing equation (10) by equation (12),

Angle $\boxed{\theta_1 = \text{Tan}^{-1}\left(\frac{S_1}{C_1}\right)}$ ... (13)

Form equations (8) and (13),

Angle $\boxed{\theta_3 = \phi - \theta_1 - \theta_2}$ ... (14)

**Q46. Solve the direct kinematics problem of the manipulator shown in figure.**




Figure

**Answer :** Nov.-11, Set-2, Q4

The coordinate link diagram for the given manipulator is shown below,

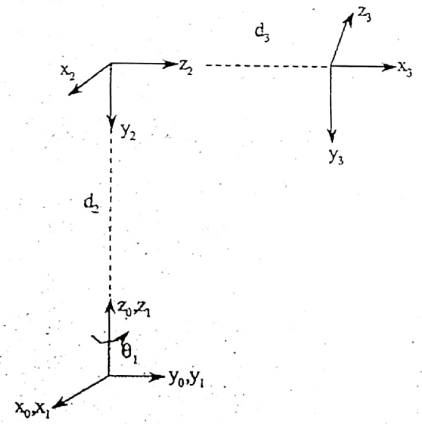



**Figure: Co-ordinate Link Diagram**

DH parameter of the given manipulator.

| Link | α | d | θ |
|---|---|---|---|
| 1 | 0 | 0 | $\theta_1$ |
| 2 | −90 | $d_2$ | 0 |
| 3 | −90 | $d_3$ | 0 |

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The transformation matrix for the given manipulator is given by,

$$T_3 = T_0 T_1 T_2$$

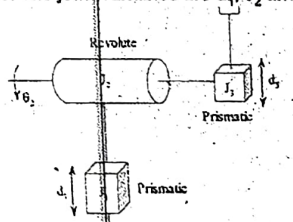
$$T_3 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0 & 0 & -C_1 & 0 \\ C_1 & 0 & -S_1 & C_1 d_3 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

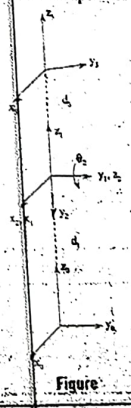
**Q47. A spatial PRP manipulator is shown in figure. Assign the link frames and obtain the D-H parameters. Write direct kinematic equations. The joint variables are $d_1$, $\theta_2$ and $d_3$.**




Figure

**Answer :** Nov.-11, Set-4, Q4

The coordinate link diagram for the given spatial PRP manipulator is shown below,




Figure

DH parameter of PRP manipulator

| Link | α | d | θ |
|---|---|---|---|
| 1 | 0 | $d_1$ | 0 |
| 2 | −90 | 0 | $\theta_2$ |
| 3 | 90 | $d_3$ | 0 |

Table

The transformation matrix for the given PRP manipulator is given by,

$$T_3 = T_0\, T_1\, T_2^*$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\alpha & \cos\theta_2(-\sin\alpha) & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & -d_3\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & d_3\cos\theta_2 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Q48. Perform the forward transformation for the five axis microbot using the following.**

| Link | a | α | θ | d |
|---|---|---|---|---|
| 1 | 0 | −90 | $\theta_1$ | $d_1$ |
| 2 | $a_2$ | 0 | $\theta_2$ | 0 |
| 3 | $a_3$ | 0 | $\theta_3$ | 0 |
| 4 | $a_4$ | 90° | $\theta_4$ | 0 |
| 5 | 0 | 0 | $\theta_5$ | $d_5$ |

**Answer :**

The general transformation matrix,

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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From the given table,

$$\text{For link 1, } {}^0T_1 = \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{For link 2, } {}^1T_2 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{For link 3, } {}^2T_3 = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_3C\theta_3 \\ S\theta_3 & C\theta_3 & 0 & a_3S\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{For link 4, } {}^3T_4 = \begin{bmatrix} C\theta_4 & S\theta_4 & 0 & a_4C\theta_4 \\ S\theta_4 & -C\theta_4 & 0 & a_4S\theta_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{For link 5, } {}^4T_5 = \begin{bmatrix} C\theta_5 & -S\theta_5 & 0 & 0 \\ S\theta_5 & C\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2 = {}^0T_1\, {}^1T_2 = \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2 = \begin{bmatrix} C\theta_1C\theta_2 & -C\theta_1S\theta_2 & -S\theta_1 & a_2C\theta_1C\theta_2 \\ S\theta_1C\theta_2 & -S\theta_1S\theta_2 & C\theta_1 & a_2S\theta_1S\theta_2 \\ -S\theta_2 & -C\theta_2 & 0 & -a_2S\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = {}^0T_2\, {}^2T_3 = \begin{bmatrix} C\theta_1C\theta_2 & -C\theta_1S\theta_2 & -S\theta_1 & a_2C\theta_1C\theta_2 \\ S\theta_1C\theta_2 & -S\theta_1S\theta_2 & C\theta_1 & a_2S\theta_1S\theta_2 \\ -S\theta_2 & -C\theta_2 & 0 & -a_2S\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_3C\theta_3 \\ S\theta_3 & C\theta_3 & 0 & a_3S\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} C_1(C_2C_3 - S_2S_3) & -C_1(C_2S_3 + S_2C_3) & -S_1 & a_3C_1(C_2C_3 - S_2S_3) + a_2C_1C_2 \\ S_1(C_2C_3 - S_2S_3) & -S_1(C_2S_3 + S_2C_3) & C\theta_1 & a_3S_1(C_2C_3 - S_2S_3) + a_2S_1S_2 \\ -(S_2C_3 + C_2S_3) & S_2S_3 - C_2C_3 & 0 & -a_3(S_2C_3 + C_2S_3) - a_2S_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_4 = {}^0T_3\, {}^3T_4$$

$${}^0T_4 = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & -S_1 & a_3C_1C_{23} + a_2C_1C_2 \\ S_1C_{23} & -S_1S_{23} & C_1 & a_3S_1C_{23} + a_2S_1S_2 \\ -S_{23} & -C_{23} & 0 & -a_3S_{23} - a_2S_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C\theta_4 & 0 & S\theta_4 & a_4C\theta_4 \\ S_4 & 0 & -C\theta_4 & a_4S\theta_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_4 = \begin{bmatrix} C_1C_{234} & -S_1 & C_1S_{234} & a_4C_1C_{234} + a_3C_1C_{23} + a_2C_1C_2 \\ S_1C_{234} & C_1 & S_1S_{234} & a_4S_1C_{234} + a_3S_1C_{23} + a_2S_1S_2 \\ -S_{234} & 0 & C_{234} & -a_4S_{234} - a_3S_{23} - a_2S_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_5 = {}^0T_4\,{}^2T_5$$

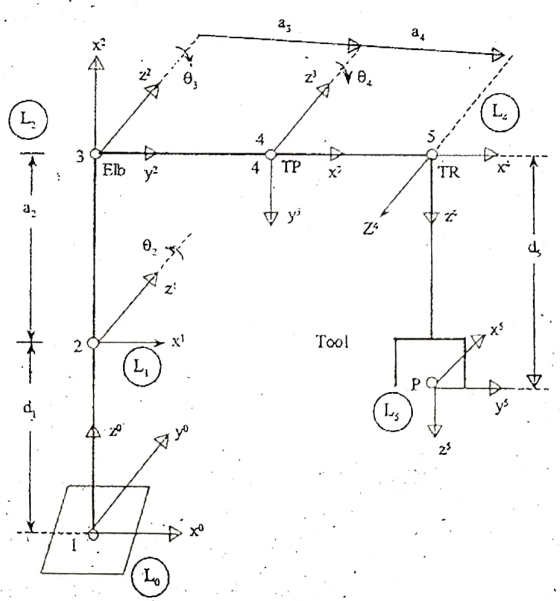
$$= \begin{bmatrix} C_1C_{234} & -S_1 & C_1S_{234} & a_4C_1C_{234} + a_3C_1C_{23} + a_2C_1C_2 \\ S_1C_{234} & C_1 & S_1S_{234} & a_4S_1C_{234} + a_3S_1C_{23} + a_2S_1S_2 \\ -S_{234} & 0 & C_{234} & -a_4S_{234} - a_3S_{23} - a_2S_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ S_5 & C_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_5 = \begin{bmatrix} C_1C_{234}C_5 - S_1S_5 & -C_1C_{234}S_5 - S_1C_5 & C_1S_{234} & C_1S_{234}d_5 + a_4C_1C_{23} + a_3C_1C_{234} + a_2C_1C_2 \\ S_1C_{234}C_5 + C_1S_5 & -S_1C_{234}S_5 + C_1C_5 & S_1S_{234} & S_1S_{234}d_5 + a_4S_1C_{234} + a_3S_1C_{23} + a_2S_1S_2 \\ -S_{234}C_5 & S_{234}S_5 & C_{234} & C_{234}d_5 - a_4S_{234} - a_3S_{23} - a_2S_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Q49. Consider the United States robots makes 110 manipulator. This is a five-axis spherical coordinate robot with a pitch-roll spherical wrist. Assign link coordinates using the first half of the D-H algorithm. Label the diagram with a's and d's as appropriate.**

**Answer :**

The five-axis spherical coordinate robot with a pitch-roll spherical wrist is used for performing successful robotic manipulation task in its work space, like pick and place. There are five joints, five-axis (three major axes-base, shoulder elbow, to position the wrist and two minor axis-pitch and roll, to orient the gripper in the direction of the object). Since, $n = 5$; 20 kinematic parameters are to be obtained and 6 unit frames are to be attached to the various joints. The link coordinate diagram of the 110 manipulator robot arm is as shown in figure.




Figure





The vector of joint variables is given by,

$q = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}^T$

The vector of joint distances are given by,

$d = \{d_1, d_2, d_3, d_4, d_5\}^T$

The vector of link lengths are given by,

$a = \{a_1, a_2, a_3, a_4, a_5\}^T$

$L_0$ to $L_5$ = Six unit frames

$d_5$ = Tool length

$q_1$ to $q_5$ = Joint variable ($q = \theta$)

$p$ = Tool tip

$d_1$ = Height of shoulder from base

1, 2, 3, 4, 5 = Rotary joints

$a_2, a_3, a_4$ = Link lengths.

| Axis | Type | $q_k$ | $d_k$ | $a_k$ | $a_k$ | SHP |
|---|---|---|---|---|---|---|
| 1 | Base | $\theta_1$ | $d_1$ | 0 | –90° | 0 |
| 2 | Shoulder | $\theta_2$ | 0 | $a_2$ | 0 | –90° |
| 3 | Elbow | $\theta_3$ | 0 | $a_3$ | 0 | 90° |
| 4 | Tool pitch | $\theta_4$ | 0 | $a_4$ | –90° | 0 |
| 5 | Tool roll | $\theta_5$ | $d_5$ | 0 | 0 | –90° |

**Table: Kinematic Parameter**

**D-H Algorithm**

To find the position $P$ and orientation $R$ of gripper with respect to base, use HCTM's starting from the tip of the gripper and ending at the base.

$$T_{base}^{tool}(q) = T_{base}^{shoul} \cdot T_{shoul}^{elbow} \cdot T_{elbow}^{pitch} \cdot T_{pitch}^{roll} \cdot T_{roll}^{TIP}$$

$$T_0^5 = T_0^1\,T_1^2\,T_2^3\,T_3^4\,T_4^5$$

$$T_0^5 = [T_0^1\,T_1^2\,T_2^3]\,[T_3^4\,T_4^5]$$

$$T_0^5 = [T_0^3\,T_5^5]$$

$$T_{base}^{tool\text{-}tip} = T_{base}^{wrist(pitch)}\;T_{wrist(pitch)}^{Tool\text{-}tip}$$

$$= f\begin{bmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \\ \theta_5 & \end{bmatrix} f\begin{bmatrix} major\ axes \\ \theta_1\ \theta_2\ \theta_3 \end{bmatrix} f\begin{bmatrix} minor\ axes \\ \theta_4\ \theta_5 \end{bmatrix}$$

$$T_0^3 = T_0^1\,T_1^2\,T_2^3 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & a_1c_1c_2 \\ s_1c_2 & -s_1s_2 & c_1 & a_1s_1s_2 \\ -s_2 & -c_2 & 0 & d_1 - a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1(c_2c_3 - s_2s_3) & -c_1(s_2c_3 + c_2s_3) & -s_1 & c_1a_3(c_2c_3 - s_2s_3) + a_2c_2 \\ s_1(c_2c_3 - s_2s_3) & -s_1(s_2c_3 + c_2s_3) & c_1 & s_1a_3(c_2c_3 - s_2s_3) + a_2c_2 \\ -(s_2c_3 + c_2s_3) & -(c_2c_3 + s_2s_3) & 1 & d_1 - a_2s_2 - a_3(s_2c_3 + c_2s_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1c_{23} & -c_1s_{23} & -s_1 & c_1(a_2c_2 + a_3c_{23}) \\ s_1c_{23} & -s_1s_{23} & c_1 & s_1(a_2c_2 + a_3c_{23}) \\ -s_{23} & -c_{23} & 0 & d_1 - a_2s_2 - a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix $T_0^3$ gives position and orientation of the wrist (PITCH) coordinate frame $L_3$ with respect to the base frame $L_0$.

$$T_3^5 = T_3^4\, T_4^5$$

$$= \begin{bmatrix} c_4 & 0 & -s_4 & a_4c_4 \\ s_4 & 0 & c_4 & a_4c_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_4c_5 & -c_4s_5 & -s_4 & a_4c_4 - d_5s_4 \\ s_4c_5 & -s_4s_5 & c_4 & a_4s_4 + d_5c_4 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix $T_3^5$ gives the position and orientation of the tip coordinate frame $L_5$ with respect to the wrist coordinate frame $L_3$.

$$T_0^5 = T_0^3\, T_3^5 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & -s_1 & c_1(a_2c_2 + a_3c_{23}) \\ s_1c_{23} & -s_1s_{23} & c_1 & s_1(a_2c_2 + a_3c_{23}) \\ -s_{23} & -c_{23} & 0 & d_1 - a_2s_2 - a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4c_5 & -c_4s_5 & -s_4 & a_4c_4 - d_5s_4 \\ s_4c_5 & -s_4s_5 & c_4 & a_4s_4 + d_5c_4 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1c_{234}c_5 + s_1s_5 & -c_1c_{234}s_5 + s_1c_5 & -c_1s_{234} & c_1(a_2c_2 + a_3c_{23} + a_4c_{234} - d_5s_{234}) \\ s_1c_{234}c_5 - c_1s_5 & -s_1c_{234}s_5 - c_1c_5 & -s_1s_{234} & s_1(a_2c_2 + a_3c_{23} + a_4c_{234} - d_5s_{234}) \\ -c_5s_{234} & s_{234}s_5 & c_{234} & d_1 - d_2s_2 - a_3s_{23} - a_4s_{234} - d_5s_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

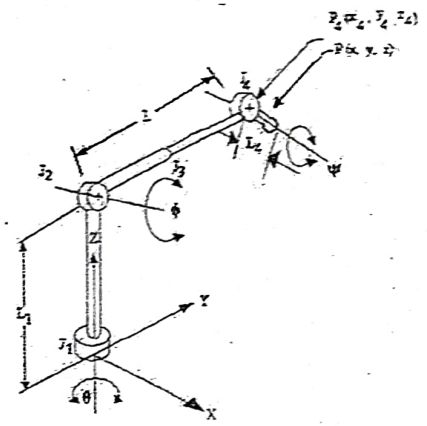
By comparing to general matrix of $T_0^5$,

$$T_0^5 = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_1 \\ R_{21} & R_{22} & R_{23} & P_2 \\ R_{31} & R_{32} & R_{33} & P_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Q50. Consider the manipulator shown in figure. Suppose the links and joints of the manipulator (had the following settings: Length of the link $L_1 = 10.0$ in, length of the extension link $L_2 = 15.0$ in, length of link 4; $L_4 = 3.0$ in, Base angle $\theta = 0°$, Elevation angle $\theta_2 = 20°$, pitch angle $\theta_4 = 34°$. Determine the coordinates of the resulting point P at which the end-of-arm would be located.**

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Figure

**Answer :**

For the above manipulator with some considerations like,

$O_0$ and $O_1$ are at same point and $L_1 = D_1 = 10$,

$O_2$ and $O_3$ are at same point and $L_2 = D_3 = 15$

The D-H parameter table can be written as,

| Link | $\theta_i$ | $d_i$ | $a_i$ | $\alpha_i$ |
|---|---|---|---|---|
| 1 | $\theta_1$ | $D_1$ | 0 | 90° |
| 2 | $\theta_2 + 90°$ | 0 | 0 | 90° |
| 3 | 0 | $D_3$ | 0 | −90° |
| 4 | $\theta_4 - 90°$ | 0 | $L_4$ | 0 |

Table

The general transformation matrix,

$$^{(i-1)}T_i = \begin{bmatrix} C\theta_i & -S\theta_iC\alpha_i & S\theta_iS\alpha_i & l_iC\theta_i \\ S\theta_i & C\theta_iC\alpha_i & -C\theta_iS\alpha_i & l_iS\theta_i \\ 0 & S\alpha_i & -C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the table the link 1 is given by,

$$^0T_1 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Considering, $\theta_1 = 0$ (given), for further simplification

For link 2,

$$^1T_2 = \begin{bmatrix} C(\theta_2 - 90°) & 0 & S(\theta_2 - 90°) & 0 \\ S(\theta_2 - 90°) & 0 & -C(\theta_2 - 90°) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_2 & 0 & C_2 & 0 \\ C_2 & 0 & S_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly, for link 3,

$$^2T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & D_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And for link 4,

$$^3T_4 = \begin{bmatrix} C(\theta_4 - 90°) & -S(\theta_4 - 90°) & 0 & L_4C(\theta_4 - 90°) \\ S(\theta_4 - 90°) & C(\theta_4 - 90°) & 0 & L_4S(\theta_4 - 90°) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_4 & C_4 & 0 & L_4S_4 \\ -C_4 & S_4 & 0 & -L_4C_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = {}^0T_1 \cdot {}^1T_2$$

$$= \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -S_2 & 0 & C_2 & 0 \\ C_2 & 0 & S_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = \begin{bmatrix} -C_1S_2 & S_1 & C_1C_2 & 0 \\ -S_1S_2 & -C_1 & S_1C_2 & 0 \\ C_2 & 0 & S_2 & D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = {}^0T_2 \cdot {}^2T_3$$

$$= \begin{bmatrix} -C_1S_2 & S_1 & C_1C_2 & 0 \\ -S_1S_2 & -C_1 & S_1C_2 & 0 \\ C_2 & 0 & S_2 & D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & D_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -C_1S_2 & -C_1C_2 & S_1 & D_3C_1C_2 \\ -S_1S_2 & -S_1C_2 & -C_1 & D_3S_1C_2 \\ C_2 & S_2 & 0 & D_3S_2 + D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_4 = {}^0T_3 \cdot {}^3T_4$$

$$= \begin{bmatrix} -C_1S_2 & -C_1C_2 & S_1 & D_3C_1C_2 \\ -S_1S_2 & -S_1C_2 & -C_1 & D_3S_1C_2 \\ C_2 & S_2 & 0 & D_3S_2 + D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_4 & C_4 & 0 & L_4S_4 \\ -C_4 & S_4 & 0 & -L_4C_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_4 = \begin{bmatrix} -S_4C_1S_2 + C_4C_1C_2 & -C_4C_1S_2 - S_4C_1C_2 & S_1 & -L_4S_4C_1S_2 + L_4C_4C_1C_2 + D_3C_1C_2 \\ -S_4S_1S_2 + C_4S_1C_2 & -C_4S_1S_2 - S_4S_1C_2 & -C_1 & -L_4S_4S_1S_2 + L_4C_4S_1C_2 + D_3S_1C_2 \\ S_4C_2 + C_4S_2 & C_4C_2 - S_4S_2 & 0 & L_4S_4C_2 + L_4C_4S_2 + D_3S_2 + D_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the above matrix for question (a) coordinates for point $P$.

Given that,

$D_1 = 10$

$D_3 = 15$

$L_4 = 3$

For $\theta_1 = 0°$, $C_1 = 1$, $S_1 = 0$

For $\theta_2 = 20°$, $C_2 = 0.9396$, $S_2 = 0.3420$

For $\theta_4 = 34°$, $C_4 = 0.829$, $S_4 = 0.5592$

$P_x = -L_4 S_4 C_1 S_2 + L_4 C_4 C_1 C_2 + D_3 C_1 C_2$

$= -3(0.5592)(1)(0.3420) + 3(0.829)(1)(0.9396) + 15(1)(0.9396)$

$= -0.5737 + 2.3367 + 14.094$

$= 15.857$ inches

$P_y = -L_4 S_4 S_1 S_2 + L_4 C_4 S_1 C_2 + D_3 S_1 C_2$

$= 0 (\because S_1 = 0)$

$P_z = L_4 S_4 C_2 + L_4 C_4 S_2 + D_3 S_2 + D_1$

$= 3(0.5592)(0.9396) + 3(0.829)(0.342) + 15(0.342) + 10$

$= 1.5763 + 0.85055 + 5.13 + 10$

$= 17.557$ inches