

UNIT-1

INTERDUCTION AND CONDUCTION HEAT TRANSFER

Heat:-

“The energy in transit is termed heat”. According to the modern or dynamical theory of heat: “Heat is a form of energy. The molecules of a substance are in parallel motion. This mean kinetic energy per molecule of the substance is proportional to its absolute temperature”. The heat is flow one region to another region due to difference in temperature. The heat flows in a direction from higher to lower temperature.

Heat transfer:-

Heat transfer is a branch of thermal science. It defined as “the transmission of energy from one region to another as a result of temperature gradient”.

The Study of heat transfer is carried out for the follows purposes:

- i. To estimate the ration of flow of energy as heat through the boundary of a system under study.
- ii. To determine the temperature field under steady and transient conditions.

Thermodynamics:-

It is a branch of thermal science that deals with the amount of heat has to be supplied or rejected from a system in order to undergo a process from one equilibrium state to another equilibrium state.

Thermodynamics, basically entails four laws or axioms known as zeroth, first, second, and third law of thermodynamics.

Zeroth law:

It deals with thermal equilibrium and establishes a concept of temperature.

First law:

The first law provides relationship between heat flow, energy stored and energy generated in a system.

Second law:

This law gives the direction of heat flow and postulates that the energy from the higher temperature to lower temperature through a system boundary.

Third law:

Third law defines absolute zero entropy.

These laws are based on experimental observations.

Thermodynamics systems:-

System:-

A system is a finite quantity of matter or a prescribed region of space.

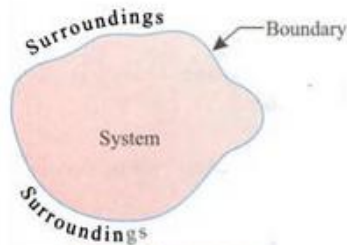


Fig. 1.1. The system.

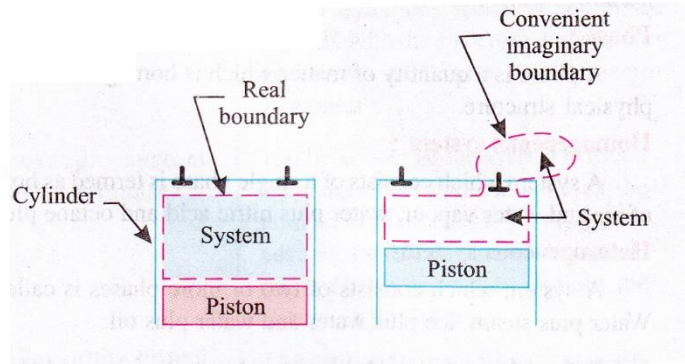


Fig. 1.2. The real and imaginary boundary.

Boundary:-

The actual or hypothetical envelop enclosing the system is the boundary of the system. The boundary may be fixed or it may move, as and when a system containing a gas is compressed or expanded. The boundary may be real or imaginary.

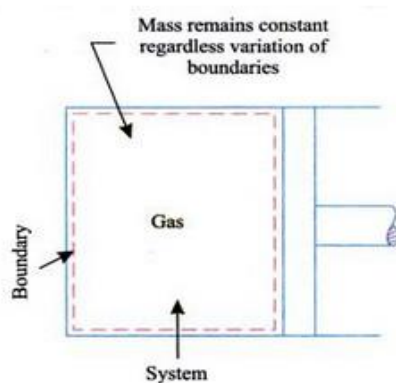


Fig. 1.3. Closed system.

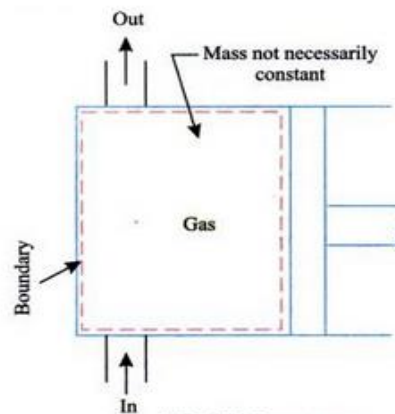


Fig. 1.4. Open system.

Open system:

An open system is one in which matter flows into or out of the system. Most of the engineering systems are open.

Isolated system:

An isolated system is that system which exchanges neither energy nor matter with any other system or with environment.

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Adiabatic system:

An adiabatic system is one which is thermally insulated from its surroundings. It can, however, exchange work with its surroundings. If it does not, it becomes an isolated system.

Phase:

A phase is a quantity of matter which is homogeneous throughout in chemical composition and physical structure.

Homogeneous system:

A system which consists of a single phase is termed as homogeneous system.

Examples: Mixture of air and water vapour, water plus nitric acid and octane plus heptane.

Heterogeneous system:

A system which consists of two or more phases is called a heterogeneous system.

Examples: Water plus steam, ice plus water and water plus oil.

Modes of heat transfer:-

Heat transfer which is defined as the transmission of energy from one region to another as a result of temperature gradient takes place by the following three modes:

(i) Conduction; (ii) Convection; (iii) Radiation.

(i) Conduction:-

"Conduction" is the transfer of heat from one part of a substance to another part of the same substance, **or** from one substance to another in physical contact with it, without appreciable displacement of molecules forming the substance.

In solids, the heat is conducted by the following two mechanisms:

(i) By lattice vibration (the faster moving molecules or atoms in the hottest part of a body transfer heat by impacts some of their energy to adjacent molecules).

(ii) By transport of free electrons (Free electrons provide an energy flux in the direction of decreasing temperature - For metals, especially good electrical conductors, the electron mechanism is responsible for the major portion of the heat flux except at low temperature).

(ii) Convection:-

"Convection" is the transfer of heat within a fluid by mixing of one portion of the fluid with another.

Generally, there are two types of convection (i) Normal (ii) Forced.

- Convection possible only in a fluid medium and is directly linked with the transport of a medium itself.
- Convection constitutes the macro form of the heat transfer since macroscopic particles of a fluid moving in space cause the heat exchange.
- The effectiveness of heat transfer by convection depends largely upon the mixing motion of the fluid.

(iii) Radiation:-

"Radiation" is the transfer of heat through space or matter by means other than conduction or convection.

- Heat is transferred from sun to earth.
- Transfer of heat from a thermos flask.

FOURIER'S LAWS OF HEAT CONDUCTION:-

Fourier's law of heat conduction is an empirical law based on observation and states as follows:

"The rate of flow of heat through a simple homogeneous solid is directly proportional to the area of the section at right angles to the direction of heat flow, and to change of temperature with respect to the length of the path of the heat flow.

Mathematically, it can be represented by the equation:

$$Q \propto A \cdot \frac{dt}{dx}$$

where,

Q = Heat flow through a body per unit time (in watts), W,

A = Surface area of heat flow (perpendicular to the direction of flow), m²,

dt = Temperature difference of the faces of block (homogeneous solid) of thickness 'dx' through which heat flows, °C or K, and

dx = Thickness of body in the direction of flow, m.

$$Q = -K \cdot A \cdot \frac{dt}{dx}$$

where,

k = Constant of proportionality and is known as thermal conductivity of the body.

The - ve sign of k is to take care of the decreasing temperature along with the direction increasing thickness or the direction of heat flow. The temperature gradient $\frac{dt}{dx}$ is always negative along positive x direction and, therefore, the value as Q become +ve.

Assumptions:

The following are the assumptions on which Fourier's law is based:

1. Conduction of heat takes place under steady state conditions.
2. The heat flow is unidirectional.
3. The temperatures gradient is constant and the temperature profile is Linear.
4. There is no internal heat generation.
5. The bounding surfaces are isothermal in character.
6. The material is homogeneous and isotropic (i.e., the value of thermal conductivity is constant in all directions).

Some essential features of Fourier's law:

Following are some essential features of Fourier's law:

1. It is applicable to all matter (may be solid, liquid or gas).
2. It is based on experimental evidence and cannot be derived from first principle.
3. It is a vector expression indicating that heat flow rate is in the direction of decreasing temperature and is normal to an isotherm.
4. It helps to define thermal conductivity ' k ' (transport property) of the medium through which heat is conducted.

THERMAL CONDUCTIVITY OF MATERIALS:-

$$k = \frac{Q}{A} \cdot \frac{dx}{dt}$$

The value of $k = 1$ when $Q = 1$, $A = 1$ and $\frac{dx}{dt} = 1$

$$\text{Now } k = \frac{Q}{A} \cdot \frac{dx}{dt} \quad (\text{unit of } k : \text{W} \times \frac{1}{\text{m}^2} \times \frac{\text{m}}{\text{K (or } ^\circ\text{C)}} = \text{W/mK. or W/m}^\circ\text{C})$$

Thus, the thermal conductivity of a material is defined as follows:

"The amount of energy conducted through a body of unit area, and unit thickness in unit time when the difference in temperature between the faces causing heat flow is unit temperature difference"

Thermal conductivity (a property of material) depends essentially upon the following factors:

- (i) Material structure (ii) Moisture content
- (ii) Density of the material (iv) Pressure and temperature (operating conditions).

THERMAL RESISTANCE (R_{th}):-

When two physical systems are described by similar equations and have similar boundary conditions, these are said to be analogous. The heat transfer processes may be compared by analogy with the flow of electricity in an electrical resistance. As the flow of electric current in the electrical resistance is directly proportional to potential difference (dV); similarly heat flow rate, Q , is directly proportional to temperature difference (dt), the driving force for heat conduction through a medium.

As per Ohm's law (in electric-circuit theory), we have

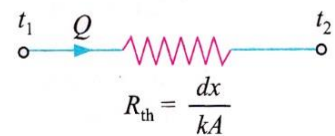
$$\text{Current } (I) = \frac{\text{Potential difference } (dV)}{\text{Electrical resistance } (R)} \quad \dots(1.4)$$

By analogy, the heat flow equation (Fourier's equation) may be written as

$$\text{Heat flow rate } (Q) = \frac{\text{Temperature difference } (dt)}{\left(\frac{dx}{kA}\right)} \quad \dots(1.5)$$

By comparing eqns. (1.4) and (1.5), we find that I is analogous to Q , dV is analogous to dt and R is analogous to the quantity $\left(\frac{dx}{kA}\right)$. The quantity $\frac{dx}{kA}$ is called **thermal conduction resistance ($R_{th})_{cond.}$** i.e.,

$$(R_{th})_{cond.} = \frac{dx}{kA}$$



- The reciprocal of the thermal resistance is called *thermal conductance*.
- It may be noted that *rules for combining electrical resistances in series and parallel apply equally well to thermal resistances*.

The concept of thermal resistance is quite helpful while making calculations for flow of heat.

HEAT TRANSFER BY CONVECTION:-

The rate equation for the convective heat transfer (regardless of particular nature) between surface and an adjacent fluid is prescribed by *Newton's law of cooling* (Refer Fig. 1.9)

$$Q = hA (t_s - t_f) \quad \dots(1.6)$$

where,

Q = Rate of conductive heat transfer,

A = Area exposed to heat transfer,

t_s = Surface temperature,

t_f = Fluid temperature, and

h = Co-efficient of convective heat transfer.

The units of h are,

$$h = \frac{Q}{A (t_s - t_f)} = \frac{W}{m^2 \text{ } ^\circ\text{C}} \text{ or } W/m^2 \text{ } ^\circ\text{C}$$

or, W/m^2K

The coefficient of convective heat transfer ' h ' (also known as *film heat transfer coefficient*) may be defined as "the amount of heat transmitted for a unit temperature difference between the fluid and unit area of surface in unit time."

The value of ' h ' depends on the following factors :

- (i) Thermodynamic and transport properties (e.g. viscosity, density, specific heat etc.).
- (ii) Nature of fluid flow.
- (iii) Geometry of the surface.
- (iv) Prevailing thermal conditions.

Since ' h ' depends upon several factors, it is difficult to frame a single equation to satisfy all the variations, however, by dimensional analysis an equation for the purpose can be obtained.

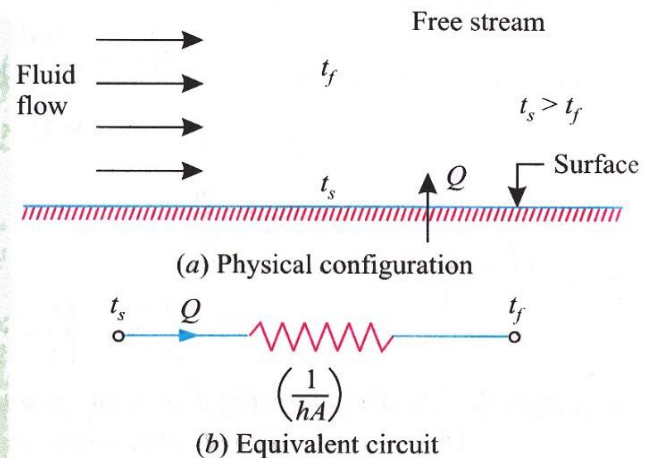


Fig. 1.9. Convective heat-transfer

The mechanisms of convection in which phase changes are involved lead to the important fields of *boiling and condensation*. Refer Fig. 1.9 (b). The quantity $\frac{1}{hA} \left[Q = \frac{t_s - t_f}{(1/hA)} \dots \text{Eqn (1.6)} \right]$ is called **convection thermal resistance** [$(R_{th})_{conv}$] to heat flow.

HEAT TRANSFER BY RADIATION:-

Laws of Radiation :

1. **Wien's law.** It states that the wavelength λ_m corresponding to the maximum energy is inversely proportional to the absolute temperature T of the hot body.

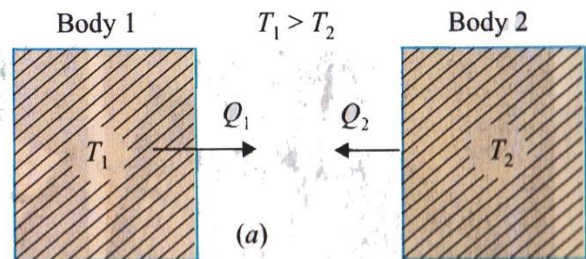
$$\text{i.e., } \lambda_m \propto \frac{1}{T} \quad \text{or, } \lambda_m T = \text{constant} \quad \dots(1.7)$$

2. **Kirchhoff's law.** It states that the emissivity of the body at a particular temperature is numerically equal to its absorptivity for radiant energy from body at the same temperature.

3. **The Stefan-Boltzmann law.** The law states that the emissive power of a black body is directly proportional to fourth power of its absolute temperature.

$$\text{i.e., } Q \propto T^4 \quad \dots(1.8)$$

Refer Fig. 1.10 (a)



$$\frac{1}{[F \sigma A (T_1 + T_2) (T_1^2 + T_2^2)]}$$

where, F = A factor depending on geometry and surface properties,

σ = Stefan-Boltzmann constant

$$= 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4,$$

A = Area, m^2 , and

T_1, T_2 = Temperatures, degrees kelvin (K).

This equation can also be rewritten as :

$$Q = \frac{T_1 - T_2}{1/[F \sigma A (T_1 + T_2) (T_1^2 + T_2^2)]} \quad \dots(1.10)$$

where denominator is **radiation thermal resistance**, $(R_{th})_{rad}$. [Fig. 1.10 (b)]

$$\text{i.e., } (R_{th})_{rad} = 1/[F \sigma A (T_1 + T_2) (T_1^2 + T_2^2)]$$

The values of F are available for simple configurations in the form of charts and tables.

$F = 1$... for simple cases of black surface enclosed by other surface

F = emissivity (ϵ) ... for non-black surface enclosed by other surface.

[Emissivity (ϵ) is defined as the ratio of heat radiated by a surface to that of an ideal surface.]

Example 1.5. A wire 1.5 mm in diameter and 150 mm long is submerged in water at atmospheric pressure. An electric current is passed through the wire and is increased until the water boils at 100°C. Under the condition if convective heat transfer coefficient is 4500 W/m²°C find how much electric power must be supplied to the wire to maintain the wire surface at 120°C?

Solution. Diameter of the wire, $d = 1.5 \text{ mm} = 0.0015 \text{ m}$

Length of the wire, $L = 150 \text{ mm} = 0.15 \text{ m}$

∴ Surface area of the wire (exposed to heat transfer),

$$A = \pi d L = \pi \times 0.0015 \times 0.15 = 7.068 \times 10^{-4} \text{ m}^2$$

Wire surface temperature, $t_s = 120^\circ\text{C}$

Water temperature, $t_f = 100^\circ\text{C}$

Convective heat transfer coefficient, $h = 4500 \text{ W/m}^2 \text{ }^\circ\text{C}$

Electric power to be supplied :

Electric power which must be supplied = Total convection loss (Q)

$$\therefore Q = hA (t_s - t_f) = 4500 \times 7.068 \times 10^{-4} (120 - 100) = \mathbf{63.6 \text{ W}}$$

Example 1.8. A surface at 250°C exposed to the surroundings at 110°C convects and radiates heat to the surroundings. The convection coefficient and radiation factor are 75W/m²°C and unity respectively. If the heat is conducted to the surface through a solid of conductivity 10W/m°C, what is the temperature gradient at the surface in the solid?

Solution. Temperature of the surface, $t_s = 250^\circ\text{C}$

Temperature of the surroundings, $t_{sur} = 110^\circ\text{C}$

The convection co-efficient, $h = 75 \text{ W/m}^2 \text{ }^\circ\text{C}$

Radiation factor, $F = 1$

Boltzmann constant, $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$

Conductivity of the solid, $k = 10 \text{ W/m}^\circ\text{C}$

Temperature gradient, $\frac{dt}{dx}$:

Heat conducted through the plate = Convection heat losses + radiation heat losses

$$\text{i.e., } Q_{cond.} = Q_{conv.} + Q_{rad.} \quad - kA \frac{dt}{dx} = hA (t_s - t_{sur}) + F\sigma A (T_s^4 - T_{sur}^4)$$

Substituting the values, we have

$$- 10 \times \frac{dt}{dx} = 75 (250 - 110) + 1 \times 5.67 \times 10^{-8} [(250 + 273)^4 - (110 + 273)^4]$$

$$\begin{aligned} - 10 \times \frac{dt}{dx} &= 10500 + 5.67 \left[\left(\frac{523}{100} \right)^4 - \left(\frac{383}{100} \right)^4 \right] \\ &= 10500 + 3022.1 = 13522.1 \end{aligned}$$

$$\therefore \frac{dt}{dx} = - \frac{13522.1}{10} = - \mathbf{1352.21 \text{ }^\circ\text{C/m}}$$

THERMAL DIFFUSIVITY:-

It is defined as the ratio of thermal conductivity to the thermal capacity. It is represented by α . Mathematically,

$$\alpha = \frac{k}{\rho c_p} \cdot m^2/s$$

Materials having higher value of ' α ' will have faster heat diffusion rate and change in temperature with time.

Thermal conductance:-

The ratio KA/L is called thermal conductance and is the reciprocal of thermal resistance. It is represented mathematically as,

$$K_{th} = \frac{KA}{L}$$

Thermal resistance:-

It is defined as the reciprocal of thermal conductance. It is denoted by R_{th} and mathematically represented as,

$$R = \frac{dx}{kA} = \frac{L}{KA}$$

It is helpful in making calculations on heat flow.

12. Thermal resistance (R_{th}) :

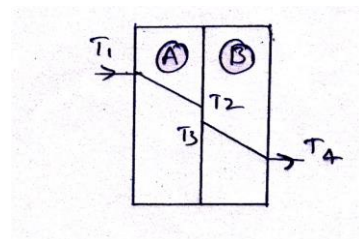
$$\text{Conduction thermal resistance, } (R_{th})_{\text{cond.}} = \frac{dx}{kA}$$

$$\text{Convection thermal resistance, } (R_{th})_{\text{conv.}} = \frac{1}{hA}$$

$$\text{Radiation thermal resistance, } (R_{th})_{\text{rad.}} = \frac{1}{1/[F \sigma A (T_1 + T_2) (T_1^2 + T_2^2)]}$$

Thermal contact resistance:-

At the contact surface of two bodies, the surface roughness and voids causes the contact surfaces to touch at discrete locations. This reduces the area of contact at the interface and results in resistance to flow of heat. This resistance is known as thermal contact resistance. Thermal contact resistance results in drop in temperature at the interface of the two materials.

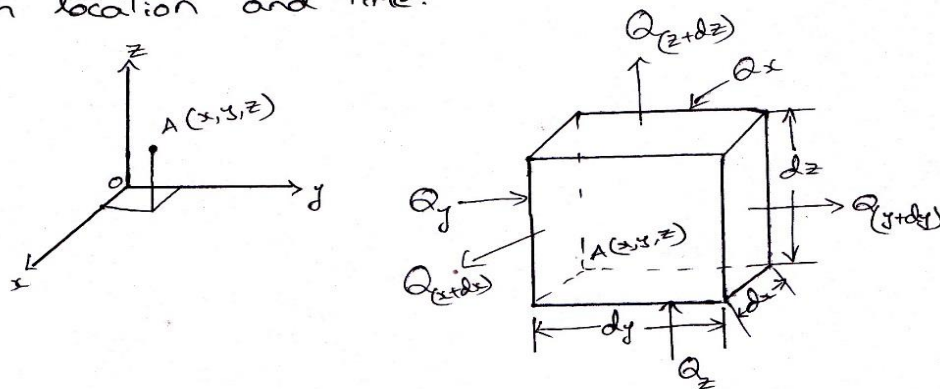


Thermal contact resistance

$$= \frac{(T_2 - T_3)}{Q/A}$$

General heat conduction equation in cartesian coordinate:-

Consider an infinitesimal rectangular parallelepiped [volume element] of sides dx , dy and dz parallel, respectively to the three axis (x, y, z) in a medium in which temp is varying with location and time.



Energy balance equation for volume element:-

rate of heat energy ^(I) conducted in + rate of heat energy generated with in the volume ^(II) element = rate of heat energy ^(III) conducted out + rate of heat energy ^(IV) accumulated with in the volume element.

$$I + II = III + IV$$

rate of heat energy conducted in ^(I) =

$$Q'_x + Q'_y + Q'_z$$

Q'_x = heat conducted into the element per unit time in x -direction.

IIIy

$Q'_y \neq Q'_z$ are heat conducted into the element per unit time in y & z-direction.

→ rate of heat energy generated within the volume element (II) = $Q'_{gen} dx dy dz$

→ rate of heat energy conducted out (III)
= $Q'_{(x+dx)} + Q'_{(y+dy)} + Q'_{(z+dz)}$

→ rate of heat energy stored within the volume element (IV) = $\rho(dx dy dz) c \frac{dT}{dt}$

where c = specific heat of the material.

ρ = mass density of material.

Now, Fourier law of heat conduction.

$$Q'_x = -k dy dz \frac{dT}{dx}, \quad Q'_y = -k dx dz \frac{dT}{dy}$$

$$Q'_z = -k dx dy \frac{dT}{dz}$$

I - III

$$= Q'_x - Q'_{(x+dx)} + Q'_y - Q'_{(y+dy)} + Q'_z - Q'_{(z+dz)}$$

$$= Q'_x - \left[Q'_x + \frac{d}{dx}(Q'_x) dx \right] + Q'_y - \left[Q'_y + \frac{d}{dy}(Q'_y) dy \right]$$

$$+ Q'_z - \left[Q'_z + \frac{d}{dz}(Q'_z) dz \right]$$

$$= \left[-\frac{d}{dx} Q'_x dx \right] + \left[-\frac{d}{dy} Q'_y dy \right] + \left[-\frac{d}{dz} Q'_z dz \right]$$

$$= -\frac{d}{dx} \left[-k dy dz \frac{dT}{dx} \right] dx + \left[-\frac{d}{dy} \left[-k dx dz \frac{dT}{dy} \right] dy \right]$$

$$+ \left[-\frac{d}{dz} \left(-k dx dy \frac{dT}{dz} \right) dz \right]$$

$$\begin{aligned}
 &= k \, dx \, dy \, dz \, \frac{\partial \tilde{T}}{\partial x} + k \, dx \, dy \, dz \, \frac{\partial \tilde{T}}{\partial y} + k \, dx \, dy \, dz \, \frac{\partial \tilde{T}}{\partial z} \\
 &= k \, dx \, dy \, dz \left(\frac{\partial \tilde{T}}{\partial x} + \frac{\partial \tilde{T}}{\partial y} + \frac{\partial \tilde{T}}{\partial z} \right) = I - III
 \end{aligned}$$

$$\text{Then } I - III + II = IV$$

$$k \, dx \, dy \, dz \left(\frac{\partial \tilde{T}}{\partial x} + \frac{\partial \tilde{T}}{\partial y} + \frac{\partial \tilde{T}}{\partial z} \right) + Q'_{\text{gen}} \, dx \, dy \, dz = \rho \, dx \, dy \, dz \, C \, \frac{dT}{dt}$$

$$\Rightarrow k \left(\frac{\partial \tilde{T}}{\partial x} + \frac{\partial \tilde{T}}{\partial y} + \frac{\partial \tilde{T}}{\partial z} \right) + Q'_{\text{gen}} = \rho C \, \frac{dT}{dt}$$

$$\nabla \tilde{T} + \frac{Q'_{\text{gen}}}{k} = \frac{1}{\alpha} \cdot \frac{dT}{dt}$$

This is known as Fourier's field equation in rectangular coordinate.
(or)

Fourier's heat 2nd law of heat conduction.

$$\text{where } \nabla \tilde{T} = \frac{\partial \tilde{T}}{\partial x} + \frac{\partial \tilde{T}}{\partial y} + \frac{\partial \tilde{T}}{\partial z}$$

$$\begin{aligned}
 \alpha &= \frac{k}{\rho C} = \frac{\text{Thermal conductivity}}{\text{Thermal capacity}} \\
 &= \text{Thermal diffusivity}
 \end{aligned}$$

Note :- volume element is homogeneous.

simplified forms of heat conduction equation in cartesian coordinates :

- (i) For the case when *no internal source of heat generation is present*, Eqn. (2.8) reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \left[\text{Unsteady state } \left(\frac{\partial t}{\partial \tau} \neq 0 \right) \text{ heat flow with no internal heat generation} \right]$$

or, $\nabla^2 t = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$ **(Fourier's equation)** ... (2.9)

- (ii) Under the situations when temperature does not depend on time, the conduction then takes place in the steady state $\left(\text{i.e., } \frac{\partial t}{\partial \tau} = 0 \right)$ and the eqn. (2.8) reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = 0$$

or, $\nabla^2 t + \frac{q_g}{k} = 0$ **(Poisson's equation)** ... (2.10)

- In the absence of internal heat generation, Eqn. (2.10) reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0$$

or, $\nabla^2 t = 0$ **(Laplace equation)** ... (2.11)

- (iii) *Steady state and one-dimensional heat transfer:*

$$\frac{\partial^2 t}{\partial x^2} + \frac{q_g}{k} = 0 \quad \dots (2.12)$$

- (iv) *Steady state, one-dimensional, without internal heat generation*

$$\frac{\partial^2 t}{\partial x^2} = 0 \quad \dots (2.13)$$

- (v) *Steady state, two dimensional, without internal heat generation*

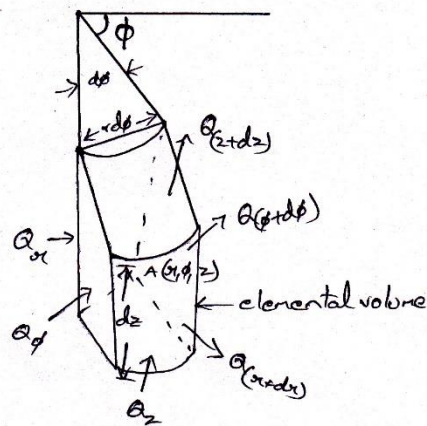
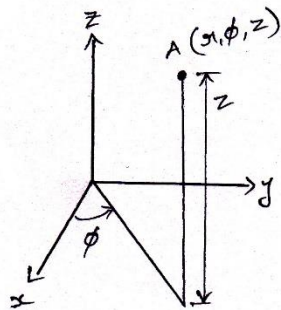
$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \quad \dots (2.14)$$

- (vi) *Unsteady state, one dimensional, without internal heat generation*

$$\frac{\partial^2 t}{\partial x^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots (2.15)$$

General heat conduction equation in cylindrical coordinates:-

Consider an elemental volume having the coordinates (r, ϕ, z) for 3-D heat conduction analysis, as shown in figure.



The volume of the element $= r dr d\phi dz$

Let Q_{gen} = heat generation (uniform) per unit volume per unit time.

Further, Let us assume that k (thermal conductivity), ρ (density), C (specific heat) do not alter with position.

$$I + II = III + IV$$

$$I = Q_r + Q_\phi + Q_z$$

$$II = Q_{gen} \times r dr d\phi dz$$

$$\text{III} = Q'_{(r+dr)} + Q'_{(\phi+d\phi)} + Q'_{(z+dz)}$$

$$\text{IV} = \rho r d\phi \cdot dr \cdot dz \cdot c \cdot \frac{dT}{dt}$$

$$Q'_r = -kr d\phi dz \frac{dT}{dr}, \quad Q'_\phi = -k dr dz \frac{dT}{r d\phi}, \quad Q'_z = -kr d\phi dr \frac{dT}{dz}$$

$$\Rightarrow \text{I} - \text{III}$$

$$\begin{aligned} &= \left(-\frac{dQ'_r}{dr} dr \right) + \left(-\frac{dQ'_\phi}{d\phi} d\phi \right) + \left(-\frac{dQ'_z}{dz} dz \right) \\ &= \frac{d}{dr} \left(kr d\phi dz \frac{dT}{dr} \right) dr + \frac{d}{d\phi} \left(k dr dz \frac{dT}{r d\phi} \right) d\phi \\ &\quad + \frac{d}{dz} \left(kr d\phi dr \frac{dT}{dz} \right) dz \end{aligned}$$

$$\begin{aligned} &= k d\phi dz dr \frac{d}{dr} \left(r \frac{dT}{dr} \right) + k d\phi dr dz \frac{d}{d\phi} \left(\frac{dT}{r d\phi} \right) \\ &\quad + kr d\phi dr dz \frac{d}{dz} \left(\frac{dT}{dz} \right) \\ &= kr d\phi dr dz \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{1}{r^2} \frac{d^2 T}{d\phi^2} + \frac{d^2 T}{dz^2} \right] \end{aligned}$$

$$\text{I} - \text{III} + \text{II} = \text{IV}$$

$$\begin{aligned} \Rightarrow kr d\phi dr dz \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{1}{r^2} \frac{d^2 T}{d\phi^2} + \frac{d^2 T}{dz^2} \right] \\ + Q'_{gen} r d\phi dz dr = r d\phi dz dr c \cdot \frac{dT}{dt} \end{aligned}$$

$$\Rightarrow \frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{1}{r^2} \frac{d^2 T}{d\phi^2} + \frac{d^2 T}{dz^2} + \frac{Q'_{gen}}{k} = \frac{1}{\alpha} \cdot \frac{dT}{dt}$$

$$\text{or,} \quad \nabla^2 T + \frac{Q'_{gen}}{k} = \frac{1}{\alpha} \cdot \frac{dT}{dt}$$

where

$$\nabla^2 = \frac{1}{r} \cdot \frac{d}{dr} \left(r \frac{d}{dr} \right) + \frac{1}{r^2} \left(\frac{d^2}{d\phi^2} \right) + \frac{d^2}{dz^2}$$

General heat conduction equation in spherical coordinates:-

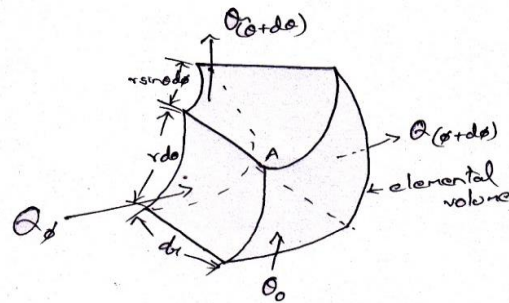
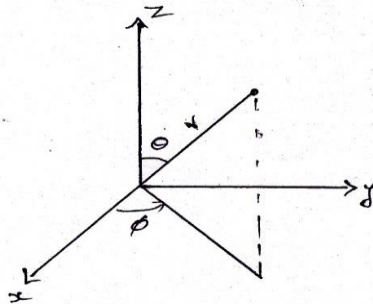
Consider an element volume having the coordinates (r, θ, ϕ) for the three dimensional heat conduction analysis, as shown in fig.

The volume of the element = $dr \cdot r d\theta \cdot r \sin\theta d\phi$

Let, Q_{gen} = heat generation per unit volume per unit time.

Further let us assume that k (thermal conductivity), ρ (density), c (specific heat) do not alter with position.

Spherical coordinate system



$$I - III + II = IV$$

$$\left(-\frac{d}{dr} Q_r'\right) dr + \left(-\frac{d}{d\theta} Q_\theta'\right) d\theta + \left(-\frac{d}{d\phi} Q_\phi'\right) d\phi \cdot r \sin\theta d\phi$$

$$Q_r' = -k r d\theta \cdot r \sin\theta d\phi \frac{dT}{dr}$$

$$Q_\theta' = -k dr \cdot r \sin\theta d\phi \frac{dT}{r d\theta}$$

$$Q_\phi' = -k dr \cdot r d\theta \frac{dT}{r \sin\theta d\phi}$$

$$II \Rightarrow Q_{gen}' dr \cdot r d\theta \cdot r \sin\theta d\phi$$

$$IV \Rightarrow dr \cdot r d\theta \cdot r \sin\theta d\phi \cdot \rho \cdot c \cdot \frac{dT}{dt}$$

$$\Rightarrow \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) + \frac{1}{r \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dT}{d\theta} \right) + \frac{1}{r \sin\theta} \frac{dT}{d\phi}$$

$$+ \frac{Q_{gen}'}{k} = \frac{1}{\alpha} \cdot \frac{dT}{dt}$$

$$\nabla^2 T + \frac{Q_{gen}'}{k} = \frac{1}{\alpha} \cdot \frac{dT}{dt}$$

$$\nabla^2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{1}{r \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right)$$

$$+ \frac{1}{r \sin\theta} \frac{d}{d\phi}$$