HIGHER ORDER ELEMENTS

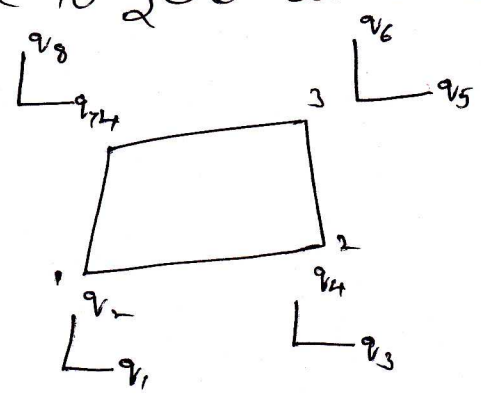
Four noded quadrilateral elements:

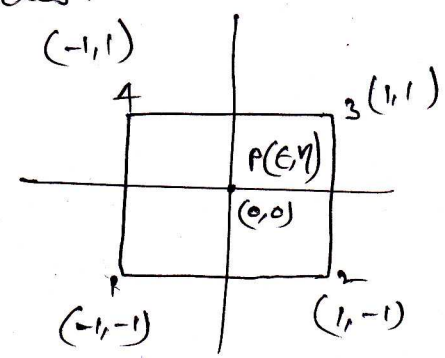
Consider the general quadrilateral element the local nodes are numbered as 1,2,3,4 in counter clock-wise direction. x_i, y_i are the coordinate of node 'i'. The vector $Q = [q_1 q_2 q_3 \cdots q_8]^T$ denotes the element displacement vector. The displacement of an interior point 'p' is located at x,y is represented as 'u'.

$$u = [u(x,y), v(x,y)]^T$$

Shape Functions:-

A master is defined in ϵ & η coordinates and is a square shaped the leglanch shape functions where $i = 1,2,3,4$ are defined such that N_i = unity at node 'i' & is equal to zero at other nodes.





In particular consider the defination of N_i ie., $N_1 = 1$ at node 1 & equal to zero at 2,3 & 4. Now the requirement that $N_1 = 0$ at nodes 2,3, & 4 is equilent to the requiring

that $N_1=0$ along the edges. $\epsilon=+1$ & $\eta=+1$. Thus N_1 has been formed

$$N_1 = c(1-\epsilon)(1-\eta)$$

where c is some constant. The constant is determined from the condition. $N_1=1$ at node '1'

$\because \epsilon=-1 \quad \eta=-1$

$\therefore$ sub the values in $N_1 = c(1-\epsilon)(1-\eta)$

$$1 = c(1+1)(1+1)$$

$$c = \frac{1}{4}$$

$$\therefore N_1 = \frac{1}{4}(1-\epsilon)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\epsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\epsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\epsilon)(1+\eta)$$

$$\therefore u = [u\ v]^T$$

where $u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$ — (A)

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8$$ — (B)

which can be written in the form of $u=Nq$

where $N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$

For isoperametric formation we use shape functions N_i to also express the coordinates of the points

Thus,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$ — (1)

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$ — (2)

Subsequently we need to express the derivatives of the function in x,y coordinates

in terms of its derivatives (ϵ, η) coordinates.

∴ This is denoted by follows.

$f = f(x,y)$ can be considered to be implecant function of ϵ & η.

∴ $f = f[x(\epsilon,\eta), y(\epsilon,\eta)]$ by using the chain rule differential eqn

$$\frac{df}{d\epsilon} = \frac{df}{dx}\frac{dx}{d\epsilon} + \frac{df}{dy}\frac{dy}{d\epsilon}$$

$$\frac{df}{d\eta} = \frac{df}{dx}\frac{dx}{d\eta} + \frac{df}{dy}\frac{dy}{d\eta}$$

$$J = \begin{bmatrix} \frac{dx}{d\epsilon} & \frac{dy}{d\epsilon} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$ where J is Jacobian matrix.

sub N_1, N_2 & N_3 & N_4 in above eqn's ① & ②

$$x = \frac{1}{4}(1-\epsilon)(1-\eta)x_1 + \frac{1}{4}(1+\epsilon)(1-\eta)x_2 + \frac{1}{4}(1+\epsilon)(1+\eta)x_3 + \frac{1}{4}(1-\epsilon)(1+\eta)x_4$$

$$\frac{dx}{d\epsilon} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

W.K.T $$\frac{dy}{d\epsilon} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$\frac{dx}{d\eta} = \frac{1}{4}\left[-(1-\epsilon)x_1 - (1+\epsilon)x_2 + (1+\epsilon)x_3 + (1-\epsilon)x_4\right]$$

$$\frac{dy}{d\eta} = \frac{1}{4}\left[-(1-\epsilon)y_1 - (1+\epsilon)y_2 + (1+\epsilon)y_3 + (1-\epsilon)y_4\right]$$

$$\therefore \begin{bmatrix} \frac{df}{d\epsilon} \\ \frac{df}{d\eta} \end{bmatrix} = J \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{df}{d\epsilon} \\ \frac{df}{d\eta} \end{bmatrix}$$

$$= \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{bmatrix} \frac{\partial f}{\partial \epsilon} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}$$

$$\therefore \quad \epsilon = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

considering $f = u$

$$\frac{\partial u}{\partial x} = J^{-1}\begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial u}{\partial y} = J^{-1}\begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

$$\text{III/y} \quad \frac{\partial v}{\partial x} = J^{-1}\begin{bmatrix} \frac{\partial v}{\partial \epsilon} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \qquad \frac{\partial v}{\partial y} = J^{-1}\begin{bmatrix} \frac{\partial v}{\partial \epsilon} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

$$\therefore \epsilon = A \begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \epsilon} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

where $A = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$

$$\begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \epsilon} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = G q$$

Sub values of N_1, N_2, N_3 & N_4 in eqn's A & B

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$$

$$u = \frac{1}{4}(1-\epsilon)(1-\eta) q_1 + \frac{1}{4}(1+\epsilon)(1-\eta) q_3 + \frac{1}{4}(1+\epsilon)(1+\eta) q_5 + \frac{1}{4}(1-\epsilon)(1+\eta) q_7$$

$$\frac{\partial u}{\partial \epsilon} = \frac{1}{4}\left[-(1-\eta) q_1 + (1-\eta) q_3 + (1+\eta) q_5 - (1+\eta) q_7\right]$$

W.r.t η $\quad \frac{\partial u}{\partial \eta} = \frac{1}{4}\left[-(1-\epsilon) q_1 + (-1)(1+\epsilon) q_3 + (1+\epsilon) q_5 + (1-\epsilon) q_7\right]$

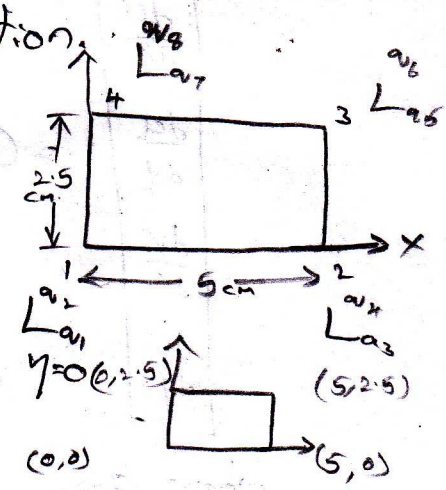
$$\frac{\partial v}{\partial \epsilon} = \frac{1}{4}\left[-(1-\eta) q_2 + (1-\eta) q_4 + (1+\eta) q_6 - (1+\eta) q_8\right]$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\left[-(1-\epsilon) q_2 - (1+\epsilon) q_4 + (1+\epsilon) q_6 + (1-\epsilon) q_8\right]$$

$\therefore$

$$G = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) \end{bmatrix}$$

← Consider the rectangular element as shown in the figure. Assume plain stress condition.



$E = 20.685 \times 10^6$ N/cm²

$\gamma = 0.3$

$q = [0, 0, 0.05, 0.075, 0.15, 0.8, 0, 0]^T$

Evaluate jacobian J, B, & σ at $\xi = 0$, $\eta = 0$

Sol:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$= \frac{1}{4}\left[-(0) + 1\times 5 + 1\times 5 - 0\right] = \frac{10}{4} = 2.5$$

$$J_{12} = \frac{1}{4}\left[-(0) + 1\times 0 + 1\times 2.5 + 1\times 2.5\right]$$

$$= 0$$

$$J_{21} = \frac{1}{4}\left[-1\times 0 - 1\times 5 + 1\times 5 + 0\right] = 0$$

$$J_{22} = \frac{1}{4}\left[-1\times(0) - 1\times 0 + 1\times 2.5 + 1\times 2.5\right] = \frac{5}{4}$$

$$= 1.25$$

$$\det J = |ad - bc| = |2.5\times 1.25 - 0|$$

$$= 3.125$$

$$A = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

$$= \frac{1}{3.125}\begin{bmatrix} 1.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 2.5 & 1.25 & 0 \end{bmatrix}_{3\times 4}$$

$$G = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 8}$$

$$B = A \times G$$

$$= \frac{1}{3.125}\begin{bmatrix} 1.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 2.5 & 1.25 & 0 \end{bmatrix}_{3\times 4} * \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 8}$$

$$= 0.08\begin{bmatrix} -1.25 & 0 & 1.25 & 0 & 1.25 & 0 & -1.25 & 0 \\ 0 & -2.5 & 0 & -2.5 & 0 & 2.5 & 0 & 2.5 \\ -2.5 & -1.25 & -2.5 & 1.25 & 2.5 & 1.25 & 2.5 & -1.25 \end{bmatrix}_{3\times 8}$$

$$B = \begin{bmatrix} -0.1 & 0 & 0.1 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.1 & 0.2 & 0.1 & 0.2 & -0.1 \end{bmatrix}_{3\times 8}$$

$$\sigma = DBq$$

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{20.685\times10^6}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 22.7307\times10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\sigma = DBq$$

$$= 22.73\times10^6\underset{3\times3}{\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}\begin{bmatrix} -0.1 & 0 & 0.1 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.1 & 0.2 & 0.1 & 0.2 & -0.1 \end{bmatrix}_{3\times}$$

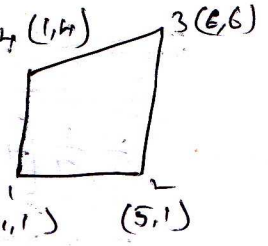
$$\begin{bmatrix} 0 \\ 0 \\ 0.05 \\ 0.07 \\ 0.15 \\ 0.8 \\ 0 \\ 0 \end{bmatrix}_{8\times1}$$

$$= 22.73\times10^6\begin{bmatrix} -0.1 & -0.06 & 0.1 & -0.06 & 0.1 & 0.06 & -0.1 & 0.06 \\ -0.03 & -0.2 & 0.03 & -0.2 & 0.03 & 0.2 & -0.03 & 0.2 \\ -0.07 & -0.035 & -0.07 & 0.035 & 0.07 & 0.035 & 0.07 & -0.035 \end{bmatrix}_{8\times8}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0.05 \\ 0.07 \\ 0.15 \\ 0.8 \\ 0 \\ 0 \end{bmatrix}_{8\times1}$$

$$= 22.73\times10^6\begin{bmatrix} 0.0638 \\ 0.152 \\ 0.0374 \end{bmatrix}_{3\times1}$$

node are given in the fig. The element displacement vector is given by as $q = [0, 0, 0.20, 0, 0.15, 0.1, 0, 0.05]$ the x,y coordinates of the point 'P' whose location in the master element is given by $\epsilon = 0.5$ & $\eta = 0.5$ & also find u, v displacement of point P



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$$

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8$$

$$N_1 = \frac{1}{4}(1-\epsilon)(1-\eta) = 0.0625$$

$$N_2 = \frac{1}{4}(1+\epsilon)(1-\eta) = 0.0187$$

$$N_3 = \frac{1}{4}(1+\epsilon)(1+\eta) = 0.562$$

$$N_4 = \frac{1}{4}(1-\epsilon)(1+\eta) = 0.1875$$

$$x = \frac{1}{4}(1-\epsilon)(1-\eta)x_1 + \frac{1}{4}(1+\epsilon)(1-\eta)x_2 + \frac{1}{4}(1+\epsilon)(1+\eta)x_3 + \frac{1}{4}(1-\epsilon)(1+\eta)x_4$$

$$= \frac{1}{4}(1-0.5)(1-0.5)1 + \frac{1}{4}\,1.5\times0.5\times5 + \frac{1}{4}\,1.5\cdot1.5\cdot6 + \frac{1}{4}\,0.5\cdot1.5\cdot1$$

$$= 4.56$$

$$y = \frac{1}{4}\,0.5\times0.5\times1 + \frac{1}{4}\,1.5\times0.5\times1 + \frac{1}{4}\,1.5\times1.5\times6 + \frac{1}{4}\times0.5\times1.5\times4$$

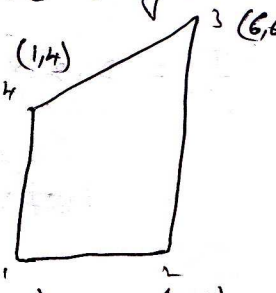
$$= 4.3$$

$$P = (x,y) = (4.56,\ 4.37)$$

$u = 0.1218$

$v = 0.065$

Using a 2×2 rule evaluate the $\iint_A (x^2 + xy^2)\,dx\,dy$ by Gaussian coordinate where 'A' denotes the region shown in the figure. The sampling points are $\pm\frac{1}{\sqrt{3}}$ & weights are unity now



~~$\epsilon = \frac{1}{\sqrt{3}}$~~ ~~$\eta = \frac{1}{\sqrt{3}}$~~

determine the jacobian for given Gaussian point.

case(i) $\epsilon = \frac{1}{\sqrt{3}}$ $\eta = \frac{1}{\sqrt{3}}$

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$= \frac{1}{4}\left[-1\left(1-\frac{1}{\sqrt{3}}\right)1 + \left(1-0.577\right)5 + (1+0.577)6 - (1+0.577)1\right]$$

$$= 2.39$$

$$J_{12} = \frac{1}{4}\left[-(1-0.577)1 + (1-0.577)1 + (1+0.577)6 - (1+0.577)4\right]$$

$$= 0.788$$

$$J_{21} = \frac{1}{4}\left[-(1-\epsilon)x_1 - (1+\epsilon)x_2 + (1+\epsilon)x_3 + (1-\epsilon)x_4\right]$$

$$= 0.394$$

$$J_{22} = 2.288$$

$$A_1 = \frac{1}{2}\left|(2.39 \times 2.288) - (0.788 \times 0.394)\right|$$

$$= 2.57$$

case(ii) $\epsilon = \frac{1}{\sqrt{3}}$ $\eta = -\frac{1}{\sqrt{3}}$

$$J_{11} = \frac{1}{4}\left[-1\left(1+\frac{1}{\sqrt{3}}\right)1 + \left(1+\frac{1}{\sqrt{3}}\right)5 + \left(1-\frac{1}{\sqrt{3}}\right)6 - \left(1-\frac{1}{\sqrt{3}}\right)1\right]$$

$$= 2.10$$

$$J_{12} = 0.211$$

$$J_{21} = \frac{1}{4}\left[-\left(1-\frac{1}{\sqrt{3}}\right)1 + \left(1+\frac{1}{\sqrt{3}}\right)5 + \left(1+\frac{1}{\sqrt{3}}\right)6 + \left(1-\frac{1}{\sqrt{3}}\right)1\right]$$

$$= 0.39$$

$$J_{22} = 2.288$$

$$A_2 = \frac{1}{2}\left|(2.10 \times 2.288) - (0.211 \times 0.39)\right|$$

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$$J_{11} = \frac{1}{4}\left[-\left(1+\frac{1}{\sqrt{3}}\right)1 + \left(1+\frac{1}{\sqrt{3}}\right)5 + \left(1-\frac{1}{\sqrt{3}}\right)6 - \left(1-\frac{1}{\sqrt{3}}\right)1\right]$$

$= 2.1$

$J_{12} = 0.2115$

$$J_{21} = \frac{1}{4}\left[-\left(1+\frac{1}{\sqrt{3}}\right)x_1 - \left(1-\frac{1}{\sqrt{3}}\right)x_2 + \left(1-\frac{1}{\sqrt{3}}\right)x_3 + \left(1+\frac{1}{\sqrt{3}}\right)x_4\right]$$

$= 0.105$

$J_{22} = 1.7115$

$A_4 = 1.78$

case (iv) $\epsilon = -\frac{1}{\sqrt{3}}$ $\eta = \frac{1}{\sqrt{3}}$

$$J_{11} = \frac{1}{4}\left[-\left(1-\frac{1}{\sqrt{3}}\right)1 + \left(1-\frac{1}{\sqrt{3}}\right)5 + \left(1+\frac{1}{\sqrt{3}}\right)6 - \left(1+\frac{1}{\sqrt{3}}\right)1\right]$$

$= 2.394$

$J_{12} = 0.7885$

$J_{21} = 0.105$

$J_{22} = 1.7115$

$A_3 = 2$

$$A = \sum_{i=1}^{4} |J| = 2.5 + 2.36 + 2 + 1.78 = 8.64$$

$$\int_A\int (x^2+xy^2)\,dx\,dy = \sum_{j=1}^{n}\sum_{i=1}^{n} w_j w_i f(x_i, y_i)\cdot dA$$

$\because w_1 = w_2$

& $\because w_1 = w_2 = 1$

$$= 8.64\left[w_1^2 f(x_1, y_1) + w_1 w_2 f(x_1, y_2) + w_2 w_1 f(x_2, y_1) + w_2^2 f(x_2, y_2)\right.$$

x_1 y_1
$(0.57735 \quad 0.57735)$
$(-0.57735 \quad -0.57735)$
x_2 y_2

$\because f = x^2+xy^2$

$$w_1^2 f(x_1, y_1) = w_1^2 (x_1^2 + x_1 y_1^2)$$

$$= 1 \left((0.57735)^2 + (0.57735)(0.57735)^2 \right)$$

$$= 0.525$$

$$w_1 w_2 f(x_1, y_2) = w_1 w_2 (x_1^2 + x_1 y_2^2)$$

$$= 1 \times 1 \left((0.57735)^2 + (0.57735)(-0.57735)^2 \right)$$

$$= 0.525$$

$$w_2 w_1 f(x_2, y_1) = w_2 w_1 (x_2^2 + x_2 y_1^2)$$

$$= 1 \times 1 \left((-0.57735)^2 + (-0.57735)(0.57735)^2 \right)$$

$$= 0.14088 \quad = 0.141$$

$$w_2^2 f(x_2, y_2) = w_2 \cdot w_2 (x_2^2 + x_2 y_2^2)$$

$$= 1 \times 1 \left((-0.57735)^2 + (-0.57735)(-0.57735)^2 \right)$$

$$= 0.14088 \quad = 0.141$$

$$\therefore \iint_A (x^2 + xy^2)\, dx\, dy = 8.64 \left[0.525 + 0.525 + 0.141 + 0.141 \right]$$

$$= 11.50$$