

CST

INTRODUCTION:-

The 2-D finite formulation in this chapter follows the steps used in the 1-D problem. The displacement, traction components and distributed body force values are functions of the position indicated by (x, y) . The displacement vector 'u' is given as:

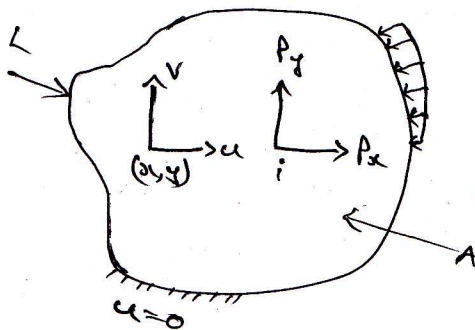
$$u = [u, v]^T$$

where u, v are the x, y components of 'u' respectively.

The stress and strains are given by

$$\sigma = [\sigma_x, \sigma_y, \tau_{xy}]^T$$

$$\epsilon = [\epsilon_x, \epsilon_y, \gamma_{xy}]^T$$



$$f = [f_x, f_y]^T, \quad T = [T_x, T_y]^T$$

The strain displacement relations are given by

$$\epsilon = \left[\frac{du}{dx}, \frac{dv}{dy}, \left(\frac{du}{dy} + \frac{dv}{dx} \right) \right]^T$$

stress and strains are related by

$$\sigma = DE.$$

ISOPARAMETRIC REPRESENTATION:-

strain-displacement matrix (B) for a 3 noded triangular element. u, v are functions of x, y & x, y are functions of ϵ, η .

$$\text{i.e., } u = N_1 q_1 + N_2 q_3 + N_3 q_5$$

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

shape functions we can write as

$$N_1 = \epsilon, \quad N_2 = \eta, \quad N_3 = 1 - \epsilon - \eta$$

$$\therefore u = \epsilon q_1 + \eta q_3 + (1 - \epsilon - \eta) q_5$$

$$u = \epsilon (q_1 - q_5) + (q_3 - q_5) \eta + q_5$$

$$v = \epsilon q_2 + \eta q_4 + (1 - \epsilon - \eta) q_6$$

$$v = (q_2 - q_6) \epsilon + (q_4 - q_6) \eta + q_6$$

The shape function matrix can be written as

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

We have

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$x = (x_1 - x_3) \epsilon + (x_2 - x_3) \eta + x_3$$

$$y = (y_1 - y_3) \epsilon + (y_2 - y_3) \eta + y_3$$

Using the notation $x_{ij} = x_i - x_j$, we can write as

$$x = x_{13} \epsilon + x_{23} \eta + x_3$$

$$y = y_{13} \epsilon + y_{23} \eta + y_3$$

we can see that $u \in v$ and $x \in y$ are the functions of e and η then

$$u = u[x(e, \eta), y(e, \eta)]$$

$$v = v[x(e, \eta), y(e, \eta)]$$

$$\frac{du}{de} = \frac{du}{dx} \cdot \frac{dx}{de} + \frac{du}{dy} \cdot \frac{dy}{de}$$

$$\frac{du}{d\eta} = \frac{du}{dx} \cdot \frac{dx}{d\eta} + \frac{du}{dy} \cdot \frac{dy}{d\eta}$$

which can be written in matrix notation as.

$$\begin{bmatrix} \frac{du}{de} \\ \frac{du}{d\eta} \end{bmatrix} = \begin{bmatrix} \frac{dx}{de} & \frac{dy}{de} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix} \begin{bmatrix} \frac{du}{dx} \\ \frac{du}{dy} \end{bmatrix}$$

where the (2×2) square matrix is denoted as the jacobian.

$$J = \begin{bmatrix} \frac{dx}{de} & \frac{dy}{de} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix}$$

on taking derivative of x and y .

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

Also

$$\begin{bmatrix} \frac{du}{dx} \\ \frac{du}{dy} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{du}{de} \\ \frac{du}{d\eta} \end{bmatrix}$$

where J^{-1} is the inverse of the jacobian J , given by.

$$J^{-1} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\det |J| = x_{13} y_{23} - x_{23} y_{13}$$

we have $A = \frac{1}{2} |J|$

$$\begin{Bmatrix} \frac{du}{dx} \\ \frac{du}{dy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} \frac{du}{d\xi} - y_{13} \frac{du}{d\eta} \\ -x_{23} \frac{du}{d\xi} + x_{13} \frac{du}{d\eta} \end{bmatrix}$$

Replacing u by the displacement v ,

$$\begin{Bmatrix} \frac{dv}{dx} \\ \frac{dv}{dy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} \frac{dv}{d\xi} - y_{13} \frac{dv}{d\eta} \\ -x_{23} \frac{dv}{d\xi} + x_{13} \frac{dv}{d\eta} \end{bmatrix}$$

Using the strain-displacement relations, we get

$$\epsilon = \begin{Bmatrix} \frac{du}{dx} \\ \frac{dv}{dy} \\ \frac{du}{dy} + \frac{dv}{dx} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} (v_1 - v_5) - y_{13} (v_3 - v_5) \\ -x_{23} (v_2 - v_6) + x_{13} (v_4 - v_6) \\ -x_{23} (v_1 - v_5) + x_{13} (v_3 - v_5) + y_{23} (v_2 - v_6) + [-y_{13} (v_4 - v_6)] \end{bmatrix}$$

From the definition of x_{ij} & y_{ij} we can

write $y_{31} = y_{13}$ & $y_{12} = y_{13} - y_{23}$ & so on.

$$\epsilon = \frac{1}{|J|} \begin{bmatrix} y_{23} v_1 + y_{31} v_3 + y_{12} v_5 \\ x_{32} v_2 + x_{13} v_4 + x_{21} v_6 \\ x_{32} v_1 + y_{23} v_2 + x_{13} v_3 + y_{31} v_4 + x_{21} v_5 + y_{12} v_6 \end{bmatrix}$$

$$\epsilon = Bq$$

where B is a (3×6) strain-displacement matrix.

$$B = \frac{1}{|J|} \begin{bmatrix} \gamma_{23} & 0 & \gamma_{31} & 0 & \gamma_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & \gamma_{23} & x_{13} & \gamma_{31} & x_{21} & \gamma_{12} \end{bmatrix}$$

$$\rightarrow D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\rightarrow \text{Element stiffness matrix } K = A \times t \times B^T \times D \times B$$

$$\rightarrow \text{stress } (\sigma) = DB\epsilon$$

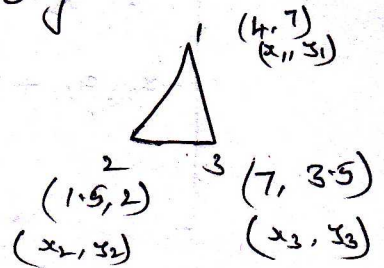
D = constant stress matrix

B = strain displacement matrix

10-11 Evaluate the shape functions N_1, N_2, N_3 at interior point 'P' for the triangular element.

Q.5.

P (3.85, 4.8)



N.K.T

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$N_1 = \epsilon \quad N_2 = \eta$$

$$N_3 = 1 - \epsilon - \eta$$

$$x = (x_1 - x_3) \epsilon + (x_2 - x_3) \eta + x_3$$

$$y = (y_1 - y_3) \epsilon + (y_2 - y_3) \eta + y_3$$

$$3.85 = -3\epsilon - 5.5\eta + 7$$

$$4.8 = 3.5\epsilon - 1.5\eta + 3.5$$

$$-3\epsilon - 5.5\eta = -3.15 \quad \text{--- ①}$$

$$3.5\epsilon - 1.5\eta = 1.3 \quad \text{--- ②}$$

∴ from eqn ① & ②

$$\epsilon = 0.5 \quad \eta = 0.3$$

$$N_1 = 0.5$$

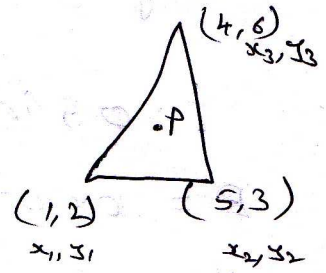
$$N_2 = 0.3$$

$$N_3 = 1 - \epsilon - \eta$$

$$= 1 - 0.5 - 0.3$$

$$= 0.2$$

element are shown in a fig at the interior point at x-coordinate is 3.3 & $N_1 = 0.3$. Determine N_2 & N_3 & coordinate point at p.



W.K.T $N_1 = \epsilon$ $N_2 = \eta$
 $= 0.3$

$$x = 3.3$$

$$x = (x_1 - x_3)\epsilon + (x_2 - x_3)\eta + x_3$$

$$3.3 = (1 - 4)0.3 + (5 - 4)\eta + 4$$

$$3.3 = -0.9 + \eta + 4$$

$$\eta = 3.3 + 0.9 - 4$$

$$\boxed{\eta = 0.2}$$

$$y = (y_1 - y_3)\epsilon + (y_2 - y_3)\eta + y_3$$

$$= (2 - 6)0.3 + (3 - 6)0.2 + 6$$

$$= -1.2 - 0.6 + 6$$

$$= 4.2$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= (0.3 \times 2) + (0.2 \times 3) + (0.5 \times 6)$$

$$= 0.6 + 0.6 + 3$$

$$= 4.2$$

$$N_3 = 1 - \epsilon - \eta$$

$$= 1 - 0.3 - 0.2$$

$$= 0.5$$

$$\therefore N_2 = 0.2 = \eta$$

$$\therefore N_3 = 0.5$$

$$\therefore p = (x, y)$$

$$= (3.3, 4.2)$$

* The 2-D loaded plate as shown in the figure. Determine the displacements of nodes 1 & 2 and element stresses using the plane stress conditions. The body force may be neglected.

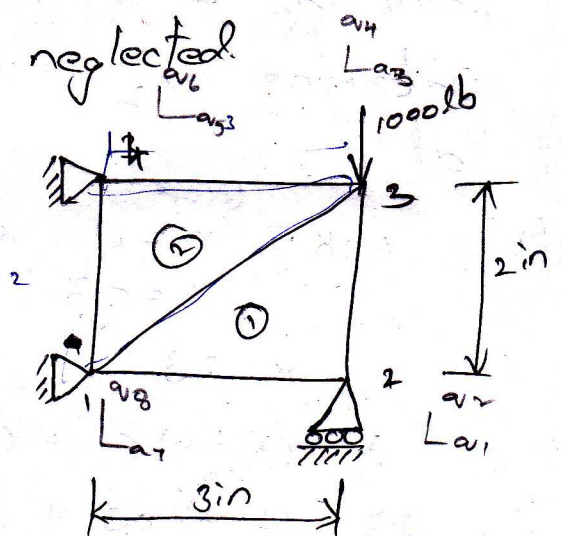
$$t = 0.5 \text{ in}$$

$$E = 30 \times 10^6 \text{ PSI}$$

$$\nu = 0.25$$

S.no:-	nodes	element
1	1-2-4	1
2	2-4-3	2

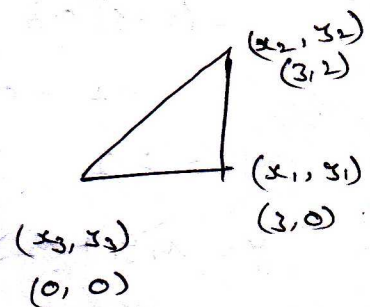
Coordinates
1 - (0,0)



Sol:-

$$B = \frac{1}{|J|} \begin{bmatrix} y_{23} & 0 & x_{32} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3 \times 6}$$

$$B^T = \frac{1}{|J|} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}_{6 \times 3}$$



$$= \frac{1}{6} \begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6 \times 3}$$

$$D = \frac{30 \times 10^6}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1 - 0.25^2}{2} \end{bmatrix}$$

$$\begin{bmatrix} 8 & 32 & 0 \\ 0 & 0 & 12096 \end{bmatrix}_{3 \times 3}$$

$$\text{Area} = \frac{1}{2} |J|$$

$$= \frac{1}{2} |x_{13} y_{23} - y_{13} x_{23}|$$

$$= \frac{1}{2} |3 \times 2 - 0|$$

$$= \frac{1}{2} (6)$$

$$= 3$$

$$t = 0.5$$

$$K_1 = t \times A \times B^T \times D \times B$$

$$= 0.5 \times 3 \times \frac{1}{6} \begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6 \times 3} \times 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12096 \end{bmatrix}_{3 \times 3}$$

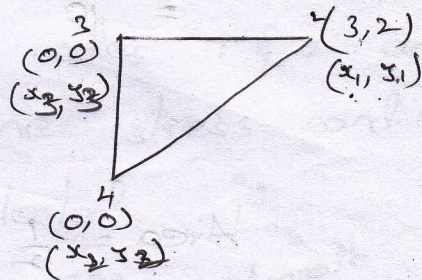
$$\times \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 \\ -2 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

$$= 41666.66 \begin{bmatrix} 64 & 16 & -24.192 \\ -16 & -64 & -24.192 \\ 0 & 0 & 36.288 \\ 24 & 96 & 0 \\ -64 & -16 & 0 \\ 0 & 0 & -24.192 \end{bmatrix}_{6 \times 3} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 \\ -2 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}_{3 \times 6}$$

$$= 41666.66 \begin{bmatrix} 176.384 & -80.384 & -72.576 & 48 & -128 & -48.384 \\ 16.384 & 79.616 & -72.576 & -192 & 48 & 48.384 \\ -72.576 & 72.576 & 108.864 & 0 & 0 & -72.576 \\ 48 & -192 & 0 & 288 & -48 & 0 \\ -128 & 32 & 0 & -48 & 128 & 0 \\ 48.384 & -48.384 & 72.576 & 0 & 0 & 48.384 \end{bmatrix}_{6 \times 6}$$

$$B^T = \frac{1}{|J|}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & -2 \\ 2 & 0 & -3 \\ 0 & -3 & 2 \end{bmatrix}$$



$$D = 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix}_{3 \times 3}$$

$$x_{13} y_{23} - y_{13} x_{23}$$

$$3 \times 0 - 2 \times 0$$

$$E = 205 \times 10^3 \text{ N/mm}^2$$

$$\nu = 0.3$$

$$K = t \times A \times B^T D B$$

$$t = 10$$

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{205 \times 10^3}{1 - (0.3)^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$= 225.2 \times 10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.355 \end{bmatrix}$$

$$\det |J|$$

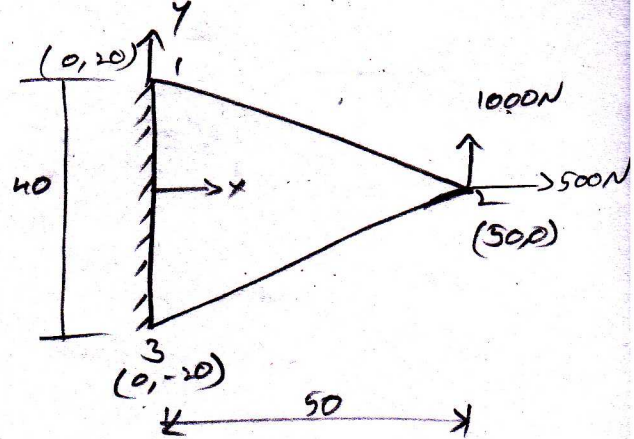
$$= |x_{13} y_{22} - x_{22} y_{13}|$$

$$= |0 \times 0 - (50 \times 40)|$$

$$= 2000$$

$$B = \frac{1}{|J|} \begin{bmatrix} y_{22} & 0 & y_{21} & 0 & y_{12} & 0 \\ 0 & x_{22} & 0 & x_{12} & 0 & x_{21} \\ x_{22} & y_{22} & x_{12} & y_{21} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{2000} \begin{bmatrix} 20 & 0 & -40 & 0 & 20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & 20 & 0 & -40 & 50 & 20 \end{bmatrix}$$



$$K = t \times A \times B^T D B$$

$$= 10 \times \frac{1}{2} \times 2000 \times \frac{1}{2000}$$

$$\begin{bmatrix} 0 & -50 & 20 \\ -40 & 0 & 0 \\ 0 & 0 & -40 \\ 20 & 0 & 50 \\ 0 & 50 & 20 \end{bmatrix}_{6 \times 3}$$

$$225 \cdot 2 \times 10^3 \begin{bmatrix} 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}_{3 \times 3}$$

$$\times \frac{1}{2000} \begin{bmatrix} 20 & 0 & -40 & 0 & 20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & 20 & 0 & -40 & 50 & 20 \end{bmatrix}_{3 \times 6}$$

$$= 563 \begin{bmatrix} 20 & 6 & -17.75 \\ -15 & -50 & 7.0 \\ -40 & -12 & 0 \\ 0 & 0 & -14.2 \\ 20 & 6 & 17.75 \\ 15 & 50 & 7.0 \end{bmatrix}_{6 \times 3}$$

$$\begin{bmatrix} 20 & 0 & -40 & 0 & 20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & 20 & 0 & -40 & 50 & 20 \end{bmatrix}_{3 \times 6}$$

$$= 563 \begin{bmatrix} 12.75 & -650 & -800 & 700 & -487.5 & -50 \\ -650 & 2640 & 600 & 280 & 50 & -2360 \\ -800 & 600 & 1600 & 0 & -800 & -600 \\ 700 & -280 & 0 & 560 & -700 & -280 \\ -487.5 & 50 & -800 & -700 & 1275 & 650 \\ -50 & -2360 & -600 & -280 & 650 & 2640 \end{bmatrix}_{6 \times 6}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 500 \\ 1000 \\ 0 \\ 0 \end{bmatrix}$$

$$563 \begin{bmatrix} 1600 & 0 \\ 0 & 560 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$\begin{bmatrix} 900800 & 0 \\ 0 & 315280 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$900800 a_3 = 500$$

$$315280 a_4 = 1000$$

$$a_3 = 5.5506 \times 10^{-4}$$

$$a_4 = 3.1717 \times 10^{-3}$$

$$t = 0.5 \text{ in}$$

$$\gamma = 0.25$$

$$E = 30 \times 10^6 \text{ PSI}$$

no. of nodes = 4

no. of elements = 2

5.10 nodes element

1	1-2-4	1
2	2-4-3	2

Coordinates

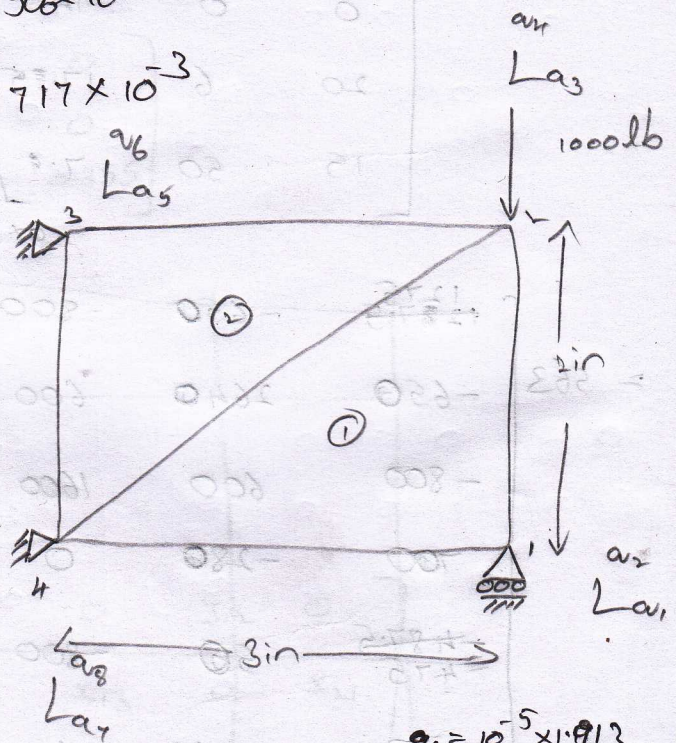
1	(0,0)
2	(0,4)
3	(3,2)
4	(3,0)

Boundary conditions:-
point load at node a_4

no. of fixed nodes

a_2, a_5, a_6, a_7, a_8

unknown values = a_1, a_3, a_4



$$a_1 = 10^{-5} \times 1.913$$

$$a_3 = 0.875 \times 10^{-5}$$

$$a_4 = -7.436 \times 10^{-5}$$

$$B^T = \begin{vmatrix} 1 \\ -3 \times 2 & -0 \end{vmatrix}$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6 \times 3}$$

$$D = \frac{30 \times 10^6}{1 - 0.25} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12096 \end{bmatrix}$$

$$K_1 = 6 \times A \times B^T \times D \times B$$

$$= 0.5 \times \frac{3}{6} \times \frac{1}{6} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6 \times 3} \times 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12096 \end{bmatrix} \times \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & 2 & -3 & 0 & 0 & -2 \end{bmatrix}_{3 \times 6}$$

$$= 41666.66 \begin{bmatrix} 64 & 16 & 36.288 \\ 24 & 96 & 24.192 \\ 0 & 0 & -36.288 \\ -24 & -96 & 0 \\ -64 & -16 & 0 \\ 0 & 0 & -24.192 \end{bmatrix}_{6 \times 3} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & 2 & -3 & 0 & 0 & -2 \end{bmatrix}_{3 \times 6}$$

$$0.0416 \times 10^6 \begin{bmatrix} 236.864 & 120.576 & -108.864 & -48 & -128 & -72.576 \\ 120.576 & 336.384 & -72.576 & -288 & -48 & -48.384 \\ -108.864 & -72.576 & +108.864 & 0 & 0 & 72.576 \\ -48 & -288 & 0 & 288 & 48 & 0 \\ -128 & -48 & 0 & 48 & 128 & 0 \\ -72.576 & -48.384 & 72.576 & 0 & 0 & 48.384 \end{bmatrix}$$

$$R_2 = t \times A \times B^T \times D \times B$$

$$= 0.5 \times \frac{9}{2} \times \frac{1}{6} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

$$10^6 \times \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12096 \end{bmatrix}$$

$$\times \frac{1}{6} \begin{bmatrix} -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 & 3 & 2 \end{bmatrix}$$

$$= 10^6 \times 0.0625 \begin{bmatrix} -96 & -24 & 0 \\ 0 & 0 & -36.288 \\ 0 & 0 & -36.288 \\ -24 & -96 & 0 \\ 64 & 16 & 36.288 \\ 24 & 96 & 24.192 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 & 3 & 2 \end{bmatrix}$$

$$10^6 \begin{bmatrix} -6 & -1.5 & 0 \\ 0 & 0 & -2.268 \\ 0 & 0 & -2.268 \\ -1.5 & -6 & 0 \\ 4 & 1 & 2.268 \\ 1.5 & 6 & 1.512 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 & 3 & 2 \end{bmatrix}$$

$$R_2 = 10^6 \begin{bmatrix} 3 & 4 & 7 & 8 & 15 & 6 \\ 18 & 0 & 0 & 4.5 & -12 & -4.5 \\ 0 & 6.804 & 6.804 & 0 & -6.804 & -4.536 \\ 0 & 6.804 & 6.804 & 0 & -6.804 & -4.536 \\ 4.5 & 0 & 0 & 18 & -3 & -18 \\ -12 & -6.804 & -6.804 & -3 & 14.804 & 7.536 \\ 3 & 4 & 7 & 8 & 15 & 6 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_1 = 10^6$$

2	5.015	13.99	-3.019	-11.98	-1.996	-2.012
3	-4.528	-3.019	4.528	0	0	3.019
4	-1.996	-11.98	0	11.98	1.996	0
7	-5.324	-1.996	0	1.996	5.324	0
8	-3.019	-2.012	3.019	0	0	2.012

$$K = K_1 + K_2$$

$$K = 10^6$$

1	9.85	-4.528	-1.996	0	0	0	0
2	-4.528	22.528	0	0	0	0	0
3	-1.996	0	18.784	0	0	0	0
4	0	0	0	18.784	0	0	0
5	0	0	0	0	18.784	0	0
6	0	0	0	0	0	18.784	0
7	0	0	0	0	0	0	18.784
8	0	0	0	0	0	0	0

$$10^6 \begin{bmatrix} 9.85 & -4.528 & -1.996 \\ -4.528 & 22.528 & 0 \\ -1.996 & 0 & 18.784 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10,000 \end{bmatrix}$$

$$a_1 = 10^{-5} \times 1.9$$

$$a_2 = 0.87 \times 10^{-5}$$

$$a_3 = -7.436 \times 10^{-5}$$

AXIS OF SYMMETRY

INTRODUCTION:

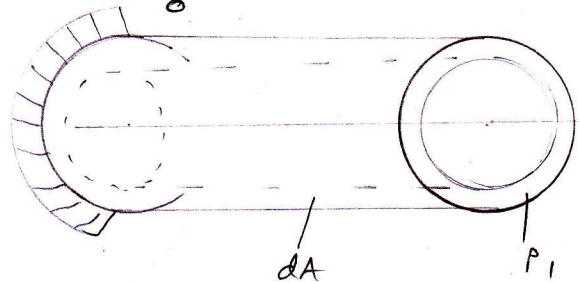
In this chapter evaluating 3-D axis of symmetry solids (or) solids of revolution subjected to axis symmetry loading reduced to simple 2-D problem. Because of total symmetric about Z-axis has seen. All the deformations and stresses are independent of rotation angle 'θ'. By considering the element volume the potential approach can be written as.

$$\Pi = \frac{1}{2} \int_0^{2\pi} \int_A \sigma^T \epsilon r d\theta dA - \int_0^{2\pi} \int_A u^T f r d\theta dA - \int_0^{2\pi} \int_A u^T T_n d\theta dA - \sum u_i P_i$$

where $u = [u, w]^T$

$f = [f_x, f_z]^T$

$T = [T_x, T_z]^T$



we can write the relation b/w strains & displacement u as $\epsilon = [\epsilon_x \ \epsilon_z \ \gamma_{xz} \ \epsilon_\theta]^T$

$$= \left[\frac{du}{dx} \quad \frac{dw}{dz} \quad \frac{du}{dz} + \frac{dw}{dx} \quad \frac{u}{r} \right]^T$$

The stress-strain relation ship is given by $\sigma = D \epsilon$

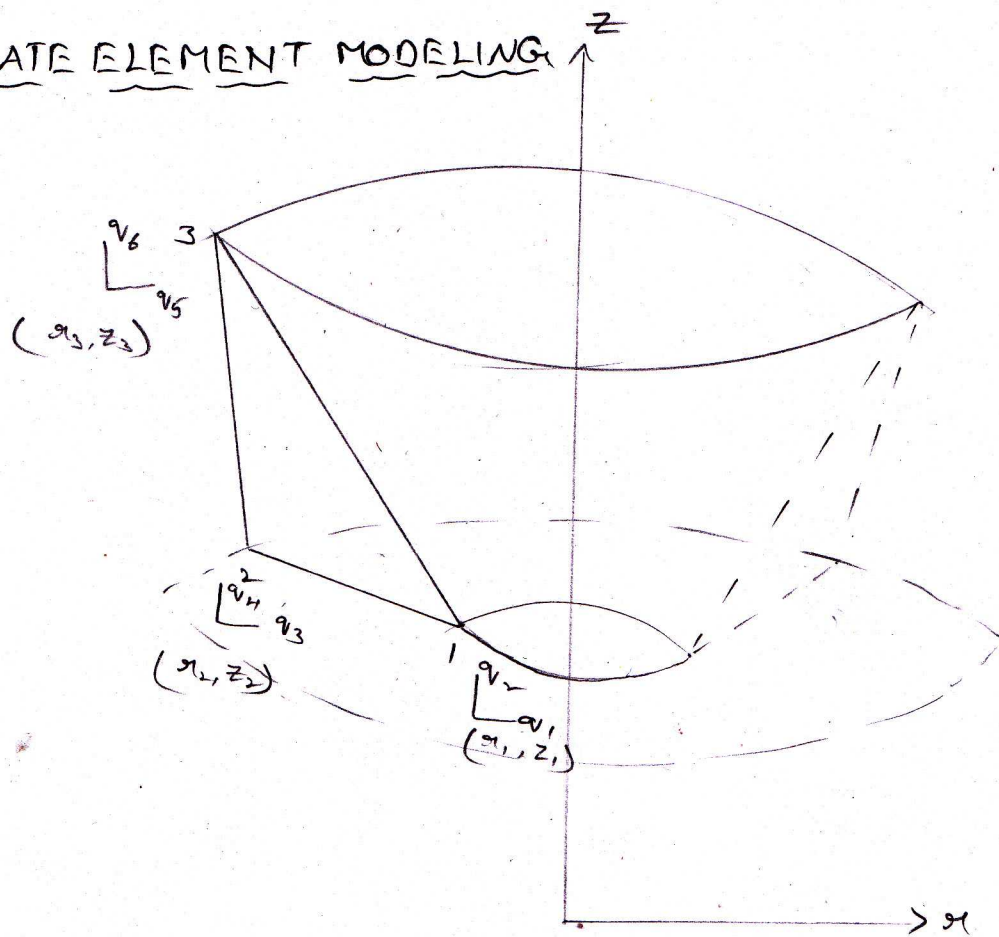
$\epsilon = B q$

$\sigma = D B q$

where D - constant stress matrix
 B - strain displacement matrix

$$\therefore D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

FINITE ELEMENT MODELING



The 2-D region defined by revolving area is divided into triangular elements. Though each element is completely represented by the area about x, z coordinates respectively. replace $x \& y$ with $x \& z$ coordinates. Using these three shape functions $N_1, N_2 \& N_3$ we define.

$$u = Nq$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$

$$w = N_1 q_2 + N_2 q_4 + N_3 q_6$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3$$

$$u = \epsilon (q_1 - q_5) + \gamma (q_3 - q_5) + q_5$$

$$w = \epsilon (q_2 - q_6) + \gamma (q_4 - q_6) + q_6$$

By using isoparametric representation we find.

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3$$

sub the values of N_1, N_2, N_3 in above eq's

$$x = \epsilon x_1 + \gamma x_2 + (1 - \epsilon - \gamma) x_3$$

$$z = \epsilon z_1 + \gamma z_2 + (1 - \epsilon - \gamma) z_3$$

The chain rule of differentiation.

$$\begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \gamma} \end{bmatrix} = J \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial z} \end{bmatrix}$$

$$\text{Similarly } \begin{bmatrix} \frac{\partial w}{\partial \epsilon} \\ \frac{\partial w}{\partial \gamma} \end{bmatrix} = J \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial z} \end{bmatrix}$$

$$\text{where } J = \begin{bmatrix} x_{1,3} & z_{1,3} \\ x_{2,3} & z_{2,3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \gamma} \end{bmatrix}$$

$$\text{Similarly } \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial w}{\partial \epsilon} \\ \frac{\partial w}{\partial \gamma} \end{bmatrix}$$

where $J^{-1} = \frac{1}{|J|} \begin{bmatrix} z_{23} & -z_{13} \\ -x_{23} & x_{13} \end{bmatrix}$

strain displacement Relations

$$\epsilon = \begin{bmatrix} \frac{du}{dx} & \frac{dw}{dz} & \frac{du}{dz} + \frac{dw}{dx} & \frac{u}{x} \end{bmatrix}^T$$

$$\epsilon = \frac{1}{|J|} \begin{bmatrix} z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_6) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6) \\ \frac{N_1 q_1 + N_2 q_3 + N_3 q_5}{x} \end{bmatrix}$$

$$\therefore B = \frac{1}{|J|} \begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{124} \\ x_{32} & z_{23} & x_{13} & z_{31} & x_{124} & z_{12} \\ \frac{N_1}{x} & 0 & \frac{N_2}{x} & 0 & \frac{N_3}{x} & 0 \end{bmatrix}$$

Element stiffness matrix

$$K = 2\pi x A B^T D B$$

where B = strain displacement matrix

D = Constant stress matrix

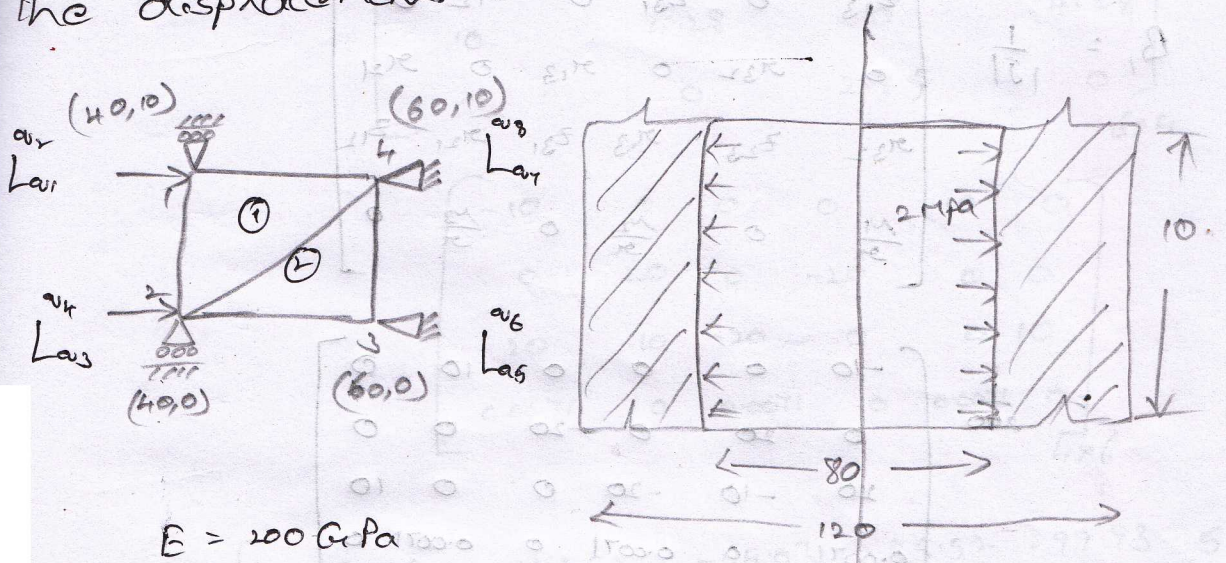
$$\bar{r} = \text{radius of Centroid} = \frac{r_1 + r_2 + r_3}{3}$$

///y shape functions

$$N_1 = N_2 = N_3 = \frac{1}{3}$$

$$\text{Area } A = \frac{1}{2} |J|$$

outside diameter 120 mm. A hole of 10 mm is drilled over its full length. The cylinder is then subjected to an internal pressure of 2 MPa using two elements on the 10 mm length. Find the displacements at the inner radius.



$$E = 200 \text{ GPa}$$

$$\nu = 0.3$$

s.no	element	nodes
1	1	1-2-4
2	2	2-3-4

coordinates

1	40, 10
2	40, 0
3	60, 0
4	60, 10

Boundary conditions

no. of fixed nodes: $u_5, u_6, v_1, v_2, v_3, v_4$

unknown values = u_1, u_3

$$\text{Area} = \frac{1}{2} |J|$$

$$= \frac{1}{2} |x_{13} z_{23} - x_{23} z_{13}|$$

$$= \frac{1}{2} |-20 \times -10 - 0|$$

$$= 100$$

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

$$= \frac{200 \times 10^3 (1-0.3)}{(1+0.3)(1-2(0.3))} \begin{bmatrix} 1 & \frac{0.3}{0.7} & 0 & \frac{0.3}{0.7} \\ \frac{0.3}{0.7} & 1 & 0 & \frac{0.3}{0.7} \\ 0 & 0 & \frac{1-0.6}{2(1-0.3)} & 0 \\ \frac{0.3}{0.7} & \frac{0.3}{0.7} & 0 & 1 \end{bmatrix}$$

$$= 2.69 \times 10^5$$

$$\begin{bmatrix} 1 & 0.428 & 0 & 0.428 \\ 0.428 & 1 & 0 & 0.428 \\ 0 & 0 & 0.285 & 0 \\ 0.428 & 0.428 & 0 & 1 \end{bmatrix}$$

$$B_1 = \frac{1}{\sqrt{1}}$$

$$\begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & z_{23} & x_{13} & z_{31} & x_{21} & z_{12} \\ \frac{N_1}{x_1} & 0 & \frac{N_2}{x_2} & 0 & \frac{N_3}{x_3} & 0 \end{bmatrix}$$

$$= \frac{1}{200}$$

$$\begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 3.55 \times 10^{-5} & 0 & 3.55 \times 10^{-5} & 0 & 3.55 \times 10^{-5} & 0 \end{bmatrix}$$

$$K_1 = 2\pi x A B^T O B$$

$$= 2\pi \times 46.66 \times 100 \times \frac{1}{200}$$

$$\begin{bmatrix} -10 & 0 & 20 & 0.0071 \\ 0 & 20 & -10 & 0 \\ 0 & 0 & -20 & 0.0071 \\ 0 & -20 & 0 & 0 \\ 10 & 0 & 0 & 0.0071 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.428 & 0 & 0.428 \\ 0.428 & 1 & 0 & 0.428 \\ 0 & 0 & 0.285 & 0 \\ 0.428 & 0.428 & 0 & 1 \end{bmatrix}$$

$$\times 2.69 \times 10^5 \times$$

$$\frac{1}{200} \begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

$$= 197.159 \times 10^{-3}$$

$$\begin{bmatrix} 8.56 & 20 & -2.85 & 8.56 \\ 3.038 \times 10^{-2} & 3.038 \times 10^{-3} & -5.7 & 0.0071 \\ -8.56 & -20 & 0 & -8.56 \\ 10 & 4.28 & 0 & 4.28 \\ 0 & 0 & 2.85 & 0 \end{bmatrix}$$

6x4

$$\begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

4x6

$$197.159 \times 10^3 \begin{bmatrix} 213.86 & -142.52 & -114.03 & 85.52 & -99.93 & 57 \\ -142.52 & 428.5 & 57 & -400 & 85.66 & -28.5 \\ -114.03 & 57 & -113.99 & -0.06 & 0.03 & -57 \\ 85.52 & -400 & -0.06 & 400 & -85.66 & 0 \\ -99.93 & 85.66 & 0.03 & -85.66 & 10.03 & 0 \\ 57 & -28.5 & -57 & 0 & 0 & 28.5 \end{bmatrix}$$

$$B_2 = \frac{1}{200} \begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \\ 6.25 \times 10^{-3} & 0 & 6.25 \times 10^{-3} & 0 & 6.25 \times 10^{-3} & 0 \end{bmatrix}$$

$$K_2 = 2\pi \times 53.33 \times 100 \times \frac{1}{200} \begin{bmatrix} -10 & 0 & 10 & 6.25 \times 10^{-3} \\ 0 & 0 & -10 & 0 \\ 10 & 0 & -20 & 6.25 \times 10^{-3} \\ 0 & -20 & 10 & 0 \\ 0 & 0 & 20 & 6.25 \times 10^{-3} \\ 0 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.428 & 0 & 0.428 \\ 0.428 & 1 & 0 & 0.428 \\ 0 & 0 & 0.85 & 0 \\ 0.428 & 0.428 & 0 & 1 \end{bmatrix}$$