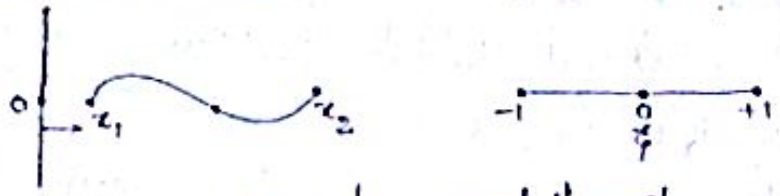
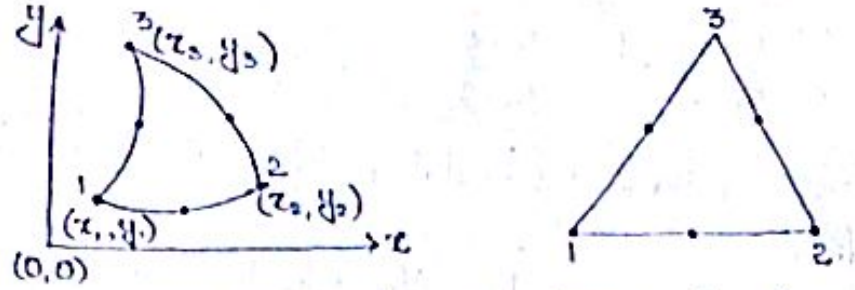
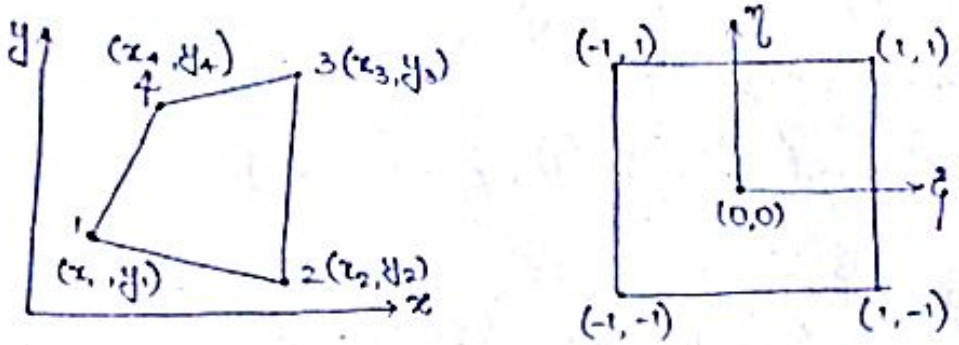
* Isoparametric element: Formulation:



(a) One dimensional element.



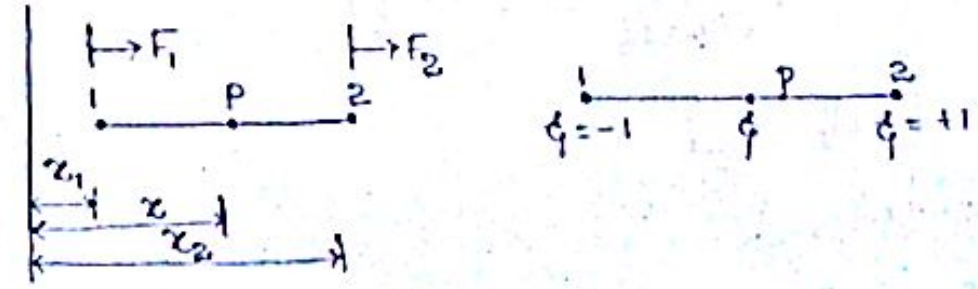
(b) Two dimensional Triangular element



(c) 2D quadrilateral element.

Here x, y = Global co-ordinate system.
ξ, η = Natural co-ordinate system.

* Consider One dimensional element :—



x_1, x_2 = Global co-ordinates of Nodes 1 & 2.

u_1, u_2 = Disp. at Nodes 1 & 2 due to Axial loads F_1 & F_2

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Then, the displacement at P is written as.

$$u = N_1 u_1 + N_2 u_2$$

For finding the value of N_1 & N_2, we use

$$u = a_1 + a_2 x.$$

by solving the above eq, using the boundary conditions we get

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \quad \& \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

And also we had the relation b/w Natural coordinate & Global coordinates

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1 \longrightarrow (1)$$

i.e, At Node 1 $x = x_1 \Rightarrow \xi = -1$

Node 2 $x = x_2 \Rightarrow \xi = +1$.

Now, we can write:

$$x = \frac{(\xi + 1)(x_2 - x_1)}{2} + x_1 \longrightarrow (2)$$

Now

$$N_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{1}{x_2 - x_1}\left[x_2 - \left[\frac{(\xi+1)(x_2 - x_1)}{2} + x_1\right]\right]$$

$$= \frac{1}{2(x_2 - x_1)}\left[2x_2 - \xi x_2 + \xi x_1 - x_2 + x_1 - 2x_1\right]$$

$$= \frac{1}{2(x_2 - x_1)}\left[(x_2 - x_1) - \xi(x_2 - x_1)\right]$$

$$= \frac{1}{2(x_2 - x_1)}(x_2 - x_1)\left[1 - \xi\right]$$

$$N_1 = \frac{1 - \xi}{2}$$

Similarly

$$N_2 = \frac{x - x_1}{x_2 - x_1} \quad \text{by solving this}$$

$$N_2 = \frac{1 + \xi}{2}$$

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$$u = \left(\frac{1-\xi}{2}\right)u_1 + \left(\frac{1+\xi}{2}\right)u_2 \longrightarrow (A)$$

Now, from eq ②

$$x = \frac{(\xi+1)(x_2-x_1)}{2} + x_1$$

$$= \frac{(\xi+1)(x_2-x_1)+2x_1}{2}$$

$$= \frac{\xi x_2 - \xi x_1 + x_2 - x_1 + 2x_1}{2}$$

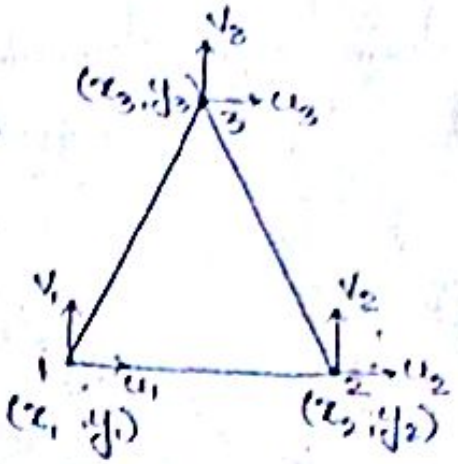
$$= \frac{\xi x_2 - \xi x_1 + x_2 + x_1}{2}$$

$$x = \left(\frac{1-\xi}{2}\right)x_1 + \left(\frac{1+\xi}{2}\right)x_2 \longrightarrow (B)$$

$\therefore x = N_1 x_1 + N_2 x_2$.

Comparing eq's A & B, that the line element displacement and geometry are described by the same shape functions in the same order. This is known as Isoparametric element.

* 2D element :-



$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

(or)

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

$N_1 = \xi \quad N_2 = \eta \quad N_3 = 1 - \xi - \eta$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

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$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$U = [N_i]_d \, q$$

Here $[N_i]_d$ = Shape function for Nodal displacements.

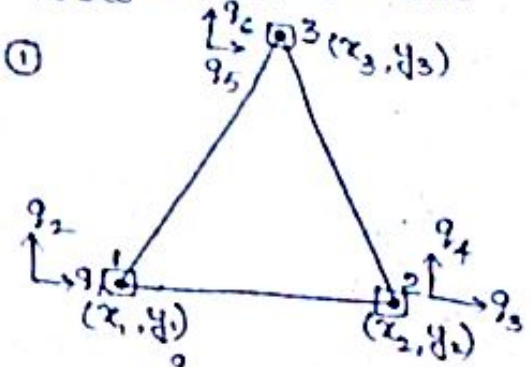
$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

$$= [N_j]_g \, C$$

$[N_j]_g$ = shape function for Geometry.

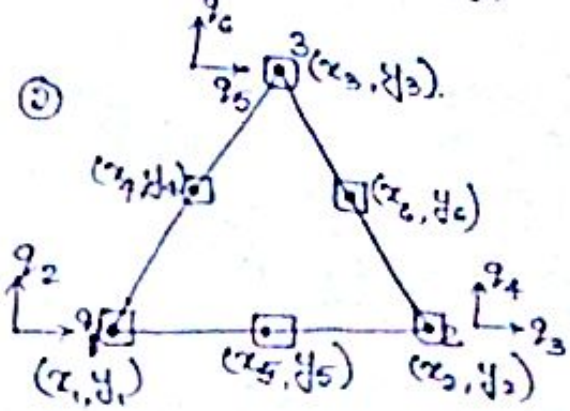
and q and C are the Nodal disp. and Nodal co-ordinate vectors.

Now for an element

① 

$[N_i]_d = [N_j]_g$

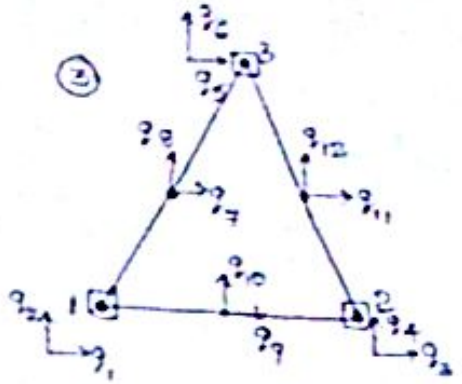
It is called Isoparametric element.

② 

$[N_i]_d < [N_j]_g$

It is called Super parametric.

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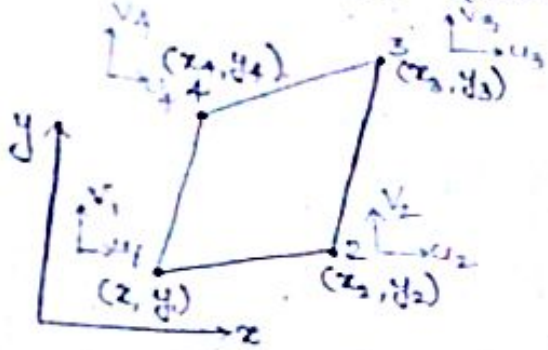
$[N_i]_d > [N_j]_g$

It is called Sub parametric.

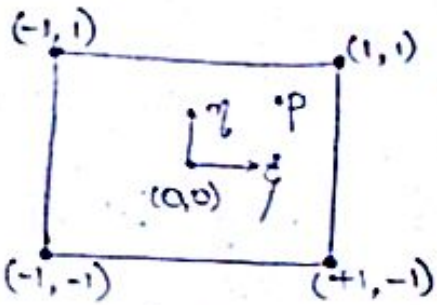
• → Nodes for defining displacement.

□ → Nodes for defining Geometry.

* Consider Four-Node Quadrilateral element :-



(a) General quadrilateral element



(b) Master (Isoparametric) element

In Isoparametric Element, all the disp. & geometry are specified by Natural coordinates ξ & η. Now, the disp. produced at any point P inside the element can be specified by the polynomial expression as.

$$u(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$\& \; v(\xi,\eta) = a_5 + a_6\xi + a_7\eta + a_8\xi\eta.$$

Let, At Node 1, $\xi = -1$: $\eta = -1$ and $u = u_1$

At Node 2, $\xi = 1$: $\eta = -1$ and $u = u_2$.

At Node 3, $\xi = 1$; $\eta = 1$ and $u = u_3$

At Node 4, $\xi = -1$; $\eta = 1$ and $u = u_4$.

By Substituting the boundary conditions in the above eq and Solving the Eq's, we get

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$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

Similarly, when solving $V(\xi,\eta)$ also we get the same formulae for Shape functions.

For the Four Noded Elements, we can write

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4.$$

& $$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4.$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

* Strain - Displacement Matrix for the Four-Noded quadrilateral element :-

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

Consider the chain Rule

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

& $$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}$$

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$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

$$= [J] \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

$[J]$ = Jacobian Matrix

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where, $J_{11} = \frac{\partial x}{\partial \xi}$, $J_{12} = \frac{\partial y}{\partial \xi}$, $J_{21} = \frac{\partial x}{\partial \eta}$, $J_{22} = \frac{\partial y}{\partial \eta}$

Now, we know that

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\left[N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4\right]$$

$$= \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4$$

$$= \frac{1}{4}(-1)(1-\eta)x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1)(1+\eta)x_3 + \frac{1}{4}(-1)(1+\eta)x_4$$

$$= \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 + (-)(1+\eta)x_4\right]$$

Similarly

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

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$$J_{22} = \frac{1}{4}\left[-(1-\xi_1)x_1 - (1+\xi_1)x_2 + (1+\xi_1)x_3 + (1-\xi_1)x_4\right]$$

Now

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

We know that

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

Here we can write

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

Now, to find the values of $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta}, \frac{\partial v}{\partial \xi}$.

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi}\left[N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4\right]$$

$$= \frac{1}{4}\left[-(1-\eta)u_1 + (1-\eta)u_2 + (1+\eta)u_3 - (1+\eta)u_4\right]$$

Similarly

$$\frac{\partial u}{\partial \eta} = \frac{1}{4}\left[-(1-\xi_1)u_1 - (1+\xi_1)u_2 + (1+\xi_1)u_3 + (1-\xi_1)u_4\right]$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)v_1 + (1-\eta)v_2 + (1+\eta)v_3 - (1+\eta)v_4\right]$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\left[-(1-\xi_1)v_1 - (1+\xi_1)v_2 + (1+\xi_1)v_3 + (1-\xi_1)v_4\right]$$

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$$\epsilon = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} d_{22} & -d_{12} & 0 & 0 \\ 0 & 0 & -d_{21} & d_{11} \\ -d_{21} & d_{11} & d_{22} & -d_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

Now here

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\epsilon = \frac{1}{|J|} \begin{bmatrix} d_{22} & -d_{12} & 0 & 0 \\ 0 & 0 & -d_{21} & d_{11} \\ -d_{21} & d_{11} & d_{22} & -d_{12} \end{bmatrix} \times \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\epsilon = [B][Q] = [G][H][Q]$$

B = Strain disp. Matrix = $[G][H]$

where $G = \frac{1}{|J|} \begin{bmatrix} d_{22} & -d_{12} & 0 & 0 \\ 0 & 0 & -d_{21} & d_{11} \\ -d_{21} & d_{11} & d_{22} & -d_{12} \end{bmatrix}$ &.

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$$H = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

* Stress - Strain Relationship Matrix :-

$$\sigma = DE = DB\delta$$

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \& \quad \epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

D = Stress - Strain Relationship Matrix.

Plane - Stress

$$D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane - Strain.

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$

* Element Stiffness Matrix

$$[K] = \int B^T D B \, dv$$

For Isoparametric elements

$$[K] = t \iint B^T D B \, dA = t \int\int B^T D B \, dx \, dy$$

(Global coordinate system)

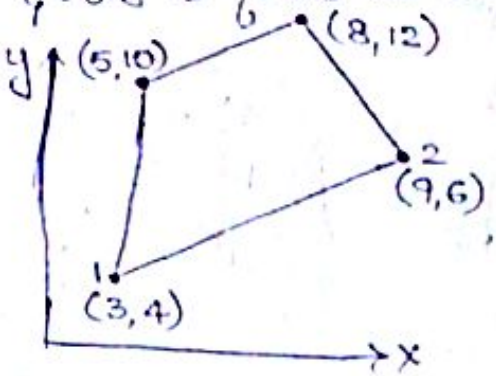
$$[K] = t \int_{-1}^{1}\int_{-1}^{1} B^T D B \, |J| \, d\xi \, d\eta$$

(For Local coordinate system).

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Problems:-

1. Determine the Cartesian co-ordinates of the point, Which has local co-ordinates $\xi = 0.8$ & $\eta = 0.6$ as shown in fig



Sol: Given

$(x_1, y_1) = (3,4)$; $(x_2, y_2) = (9,6)$: $(x_3, y_3) = (8,12)$: $(x_4, y_4) = (5,10)$

& $\xi = 0.8$, $\eta = 0.6$.

We know that

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) = \frac{1}{4}(1-0.8)(1-0.6) = 0.02.$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) = \frac{1}{4}(1+0.8)(1-0.6) = 0.18.$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) = \frac{1}{4}(1+0.8)(1+0.6) = 0.72.$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) = \frac{1}{4}(1-0.8)(1+0.6) = 0.08.$$

$\therefore x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$x = 7.84$ units.

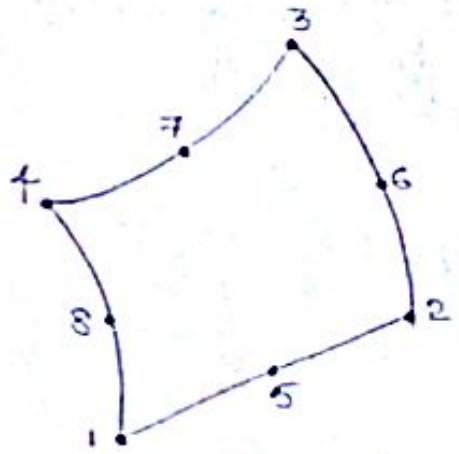
& $y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

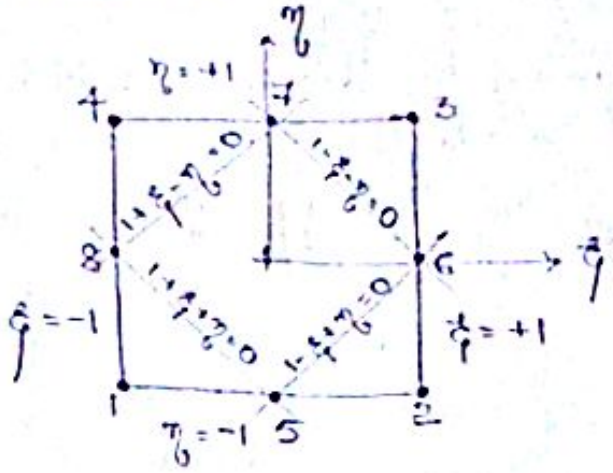
$= 10.6$ units

$\therefore$ The cartesian co-ordinates of the Points $P(x, y) = (7.84, 10.6)$

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* Eight Noded Quadrilateral Element :-





$$N_1 = -\frac{(1-\xi)(1-\eta)(1+\xi+\eta)}{4}$$

$$N_2 = -\frac{(1+\xi)(1-\eta)(1-\xi+\eta)}{4}$$

$$N_3 = -\frac{(1+\xi)(1+\eta)(1-\xi-\eta)}{4}$$

$$N_4 = -\frac{(1-\xi)(1+\eta)(1+\xi-\eta)}{4}$$

$$N_5 = \frac{(1-\xi^2)(1-\eta)}{2}$$

$$N_6 = \frac{(1+\xi)(1-\eta^2)}{2}$$

$$N_7 = \frac{(1-\xi^2)(1+\eta)}{2}$$

$$N_8 = \frac{(1-\xi)(1-\eta^2)}{2}$$

* Six Noded Element

$$N_1 = \xi(2\xi-1) \qquad N_4 = 4\xi\eta$$

$$N_2 = \eta(2\eta-1) \qquad N_5 = 4\xi\eta$$

$$N_3 = \xi(2\xi-1) \qquad N_6 = 4\xi\zeta$$

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