* **Finite Element Methods:**

The Finite Element Method is described as a Numerical Method to obtain approximate solutions to a wide variety of mostly complex problems arised in different engineering fields.

The process of Finite Element Analysis / Methods can be specified by three stages of Activities such as:

1. **Pre-processing:**
   It involves the data.
   → Nodal coordinates
   → Connectivity
   → Boundary Conditions.
   → Loading and Material information.

2. **Processing:**
   It involves the data.
   → Stiffness Generation.
   → Stiffness Modification.
   → Solution of Variables.

3. **Post Processing:** the Strain and Stress distribution, temperature etc are computed at this stage.

* **Basic Steps in Finite Element Method:**

To find the solution for a complex problem in the Finite Element method, some basic steps are followed. These steps are almost similar except slight modification in any field.

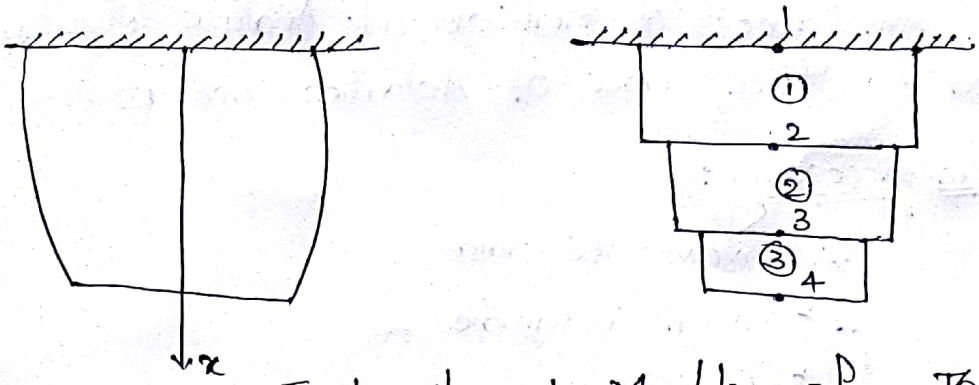
Eg: Structural Analysis, heat transfer, Fluid flow etc

The General steps involved in Structural Engineering problems are discussed as follows:

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1. Discretization of Domain:

The first step in the Finite Element Method is to divide (i.e, discretize) the whole complex structure called domain into small parts i.e, Sub-domains called Finite Elements or Lines or Surfaces. The main function of discretization of the domain is due to the following reasons.



Finite Element Modeling of a Bar.

→ In Most Engineering problems, the domains has irregular shapes. Hence the field variables used for their design such as displacements due to loading, temperature distribution due to H.T, Mass Transfer due to fluid flow etc., cannot be predicted accurately in all places by using analytical methods.

→ The irregular shapes of domain has infinite no. of degrees of freedom which results in highly complicated and time consuming computational Methods.

That is why, the irregular shaped domain is splitted into convenient no. of regular shaped sub-domains or finite elements. and thus the problem is simplified.

During discretization, the shapes, sizes, number and Configuration of the elements have to be choosen carefully.

2. Selection of displacement function:

It involves choosing a displacement function within each element. The function is defined with in the Elements using the Nodal values of the element.

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Linear, quadratic & Cubic polynomials are frequently used functions because they are simple to work.

3. Formation of Element Stiffness Matrix & Load Vector.

After discretizing the domain with desired element shapes, the element stiffness matrix and Load vector are formulated. These can be done by using either equilibrium conditions or a suitable variational principle.

4. Formation of Global Stiffness Matrix & Load Vector.

The Global Stiffness Matrix and the Global load vector are formulated from the element stiffness matrix and Element load vector.

The Final Global finite element equation for the complete structure can be written in the Matrix form as

$$[K][U] = \{F\}$$

Where, $[K]$ = Global Stiffness Matrix.

$[U]$ = Nodal displacement vector.

$\{F\}$ = Global Load Vector.

5. Incorporation / Applying Boundary Conditions:

After forming Global Finite Element Equation, the boundary conditions are imposed in that equation. The size of the stiffness matrix may be reduced and the equations required for the solution of the problem are developed. 1. Elimination Method 2. Penalty Method.

6. Solution of Simultaneous Equations:

The developed Equation (from step:5) are solved to get nodal displacements commonly by Gauss elimination method or Gauss Seidal or Jacobi Iteration Method.

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7. Calculating Strains & Stresses:

From the calculated Nodal displacements, the element strains & stresses can be computed by using the necessary equations.

8. Interpretation of the Results Obtained:

The final goal of FEA is to interpret and analyse the results obtained in the design and production process. Determining the location in the structure where large deformation and large stresses occur is generally important & essential in making design & analysis decisions.

* Coordinates and Shape Functions:

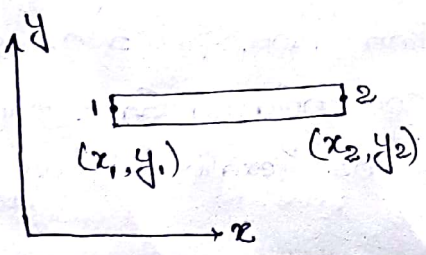
→ Co-ordinate System:

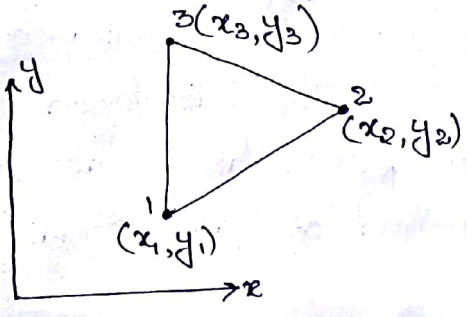
In Finite Element Method, the location of various nodes of the element must be expressed with respect to some fixed axes, for easy identification of the element and further process. These axes are called as co-ordinates, which may be placed away from the elements or on the elements themselves depending upon complexity of the problem.

In practice two types of Co-ordinate systems are adopted, they are:

1. Global Coordinate System.
2. Local Coordinate System.

* Global Coordinate System:





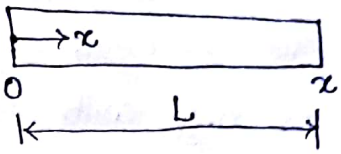
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In this system, the axis is placed completely outside the element. So, the coordinate points are represented with respect to the coordinate Axis.

* Local Coordinate System:

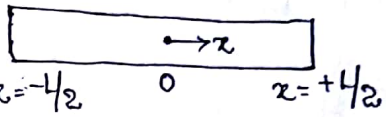
(a) Axis at the Extreme End.

Here, the axis is placed at the Extreme end as shown so that the value of x changes from 0 to L.



(b) Axis at the Mid-point:

Here, the axis is placed at the centre of the element so that the value of x changes from $-L/2$ to $L/2$.



* Natural Coordinate System:

A Natural Coordinate system is another type of Local coordinate system that permits the specification of a point within the element by a set of dimensionless numbers whose magnitude never exceeds Unity.

There are two types of Natural Coordinate system.

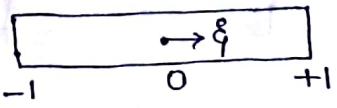
1. Axis at Extreme End

Here, the magnitude of 'ξ' varies from 0 to 1



2. Axis at Mid-point

Here, the Magnitude of 'ξ' varies from -1 to +1



Note: This Natural or Intrinsic coordinate system is denoted by ξ(xyi)

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→ **Shape Functions:**

For achieving the displacements at the Nodal points and inside the elements, we have to make use of two mathematical expressions namely.

1. Finite Element Equations
2. Shape Functions.

* **Finite Element Equation:** This equations relates the applied force with Nodal displacements, and hence by using this equations the displacements at the primary nodes (i.e., at the extreme ends of the elements) can be evaluated.

$$F = KQ$$

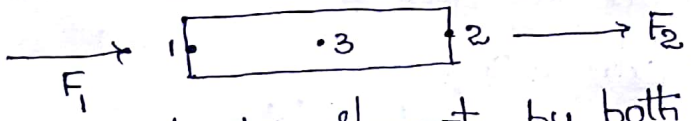
* **Shape Functions:** Shape functions are employed to find the displacements at the interior points of the element using the values of Nodal displacements. It is denoted by N.

$$u = N_1u_1 + N_2u_2 + N_3u_3 + \dots\dots$$

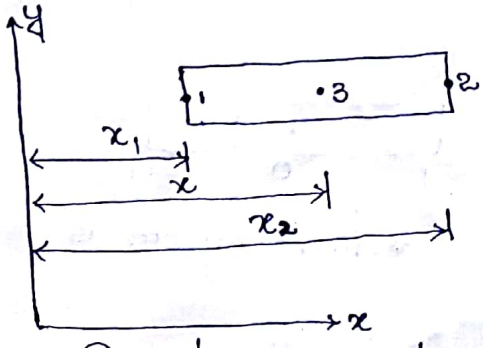
Where u_1, u_2, u_3 → Nodal displacements

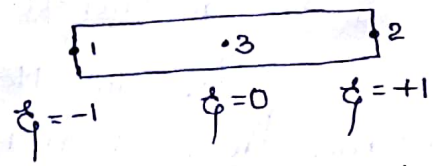
u → Interior displacement.

Consider a Bar (1D) element as shown.



Let represent the element by both the Global and Natural coordinate system.



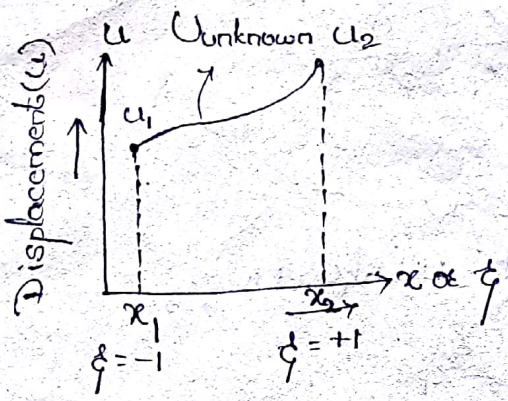


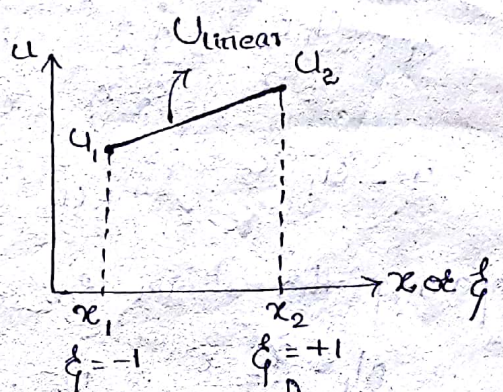
Here, x_1, x_2 & x – Global points

ξ – Natural System.

Consider the graph, the x & ξ along x axis & u along y axis.

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Interpolation function.

Now, the u can be written as a Linear polynomial function.

$u = a_1 + a_2 x$ for Absolute or Global system.

$u = a_1 + a_2 \xi$ for Natural or Intrinsic system.

(i) Consider the Absolute co-ordinate system

$$u = a_1 + a_2 x \quad \text{—(A)}$$

At Node 1 $x = x_1$, $u = u_1$

$$u_1 = a_1 + a_2 x_1 \rightarrow (1)$$

At Node 2 $x = x_2$, $u = u_2$

$$u_2 = a_1 + a_2 x_2 \quad \text{—(2)}$$

By solving (1) and (2)

$$u_1 = a_1 + a_2 x_1$$
$$u_2 = a_1 + a_2 x_2$$
$$u_1 - u_2 = a_2(x_1 - x_2)$$
$$a_2 = \frac{u_2 - u_1}{x_2 - x_1}$$

& $a_1 = u_1 - a_2 x_1$

$$= u_1 - \frac{u_2 - u_1}{x_2 - x_1} \cdot x_1 = \frac{u_1 x_2 - u_1 x_1 - u_2 x_1 + u_1 x_1}{(x_2 - x_1)}$$

$$a_1 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1}$$

Substitute in Eq. (A)

$$u = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} + \frac{u_2 - u_1}{x_2 - x_1} \cdot x$$

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$$u = \frac{u_1 x_2}{x_2 - x_1} - \frac{u_1 x}{x_2 - x_1} - \frac{u_2 x_1}{x_2 - x_1} + \frac{u_2 x}{x_2 - x_1}$$

$$u = u_1\left(\frac{x_2 - x}{x_2 - x_1}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Comparing the above eq. with $u = N_1 u_1 + N_2 u_2$

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \quad \& \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

(ii) * Consider the Natural co-ordinate system

$$u = a_1 + a_2 \xi \quad \text{---(B)}$$

At Node 1 $u = u_1$ & $\xi = -1$

$u_1 = a_1 - a_2 \rightarrow (3)$

At Node 2 $u = u_2$ & $\xi = +1$

$u_2 = a_1 + a_2 \quad \text{---(4)}$

By solving (3) & (4)

$u_1 = a_1 - a_2$

$u_2 = a_1 + a_2$

$u_1 + u_2 = 2a_1$

$\therefore a_1 = \frac{u_1 + u_2}{2}$

& $a_2 = a_1 - u_1$

$= \frac{u_1 + u_2}{2} - u_1 = \frac{u_1 + u_2 - 2u_1}{2}$

$\therefore a_2 = \frac{-u_1 + u_2}{2}$

$\therefore$ Substituting in Eq. (B)

$$u = \frac{u_1 + u_2}{2} + \frac{-u_1 + u_2}{2}\xi$$

$$= \frac{u_1}{2} - \frac{u_1}{2}\xi + \frac{u_2}{2} + \frac{u_2}{2}\xi$$

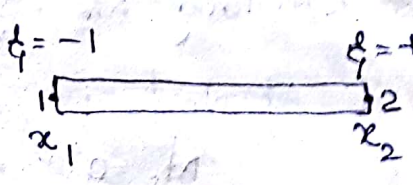
$$\therefore u = u_1\left(\frac{1 - \xi}{2}\right) + u_2\left(\frac{1 + \xi}{2}\right)$$

Again, by comparing with $u = N_1 u_1 + N_2 u_2$

$N_1 = \frac{1 - \xi}{2}$ $N_2 = \frac{1 + \xi}{2}$

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→ To find the value of ξ :-



Consider

$$\xi = a_1 + a_2 x$$

At node 1 $\xi = -1$ & $x = x_1$

$$-1 = a_1 + a_2 x_1$$

At node 2 $\xi = +1$ & $x = x_2$

$$1 = a_1 + a_2 x_2$$

Solving,

$$a_1 + a_2 x_1 = -1$$
$$a_1 + a_2 x_2 = +1$$
$$a_2(x_1 - x_2) = -2$$
$$\therefore a_2 = \frac{2}{x_2 - x_1}$$

& $a_1 = -1 - a_2 x_1$

$$\therefore a_1 = -1 - \frac{2}{x_2 - x_1} \cdot x_1$$

$$\therefore \xi = -1 - \frac{2}{x_2 - x_1} x_1 + \frac{2}{x_2 - x_1} \cdot x$$

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

We see that $\xi = -1$ at node 1 & $\xi = 1$ at node 2. The length of the element covered when ξ changes from -1 to $+1$.

* Properties of Shape function :

(1) Kronecker delta property :

The shape function has the value of 1 at that node and zero at other nodes.

Check :- Let $N_1 = \frac{x_2 - x}{x_2 - x_1}$

At node 1 $\Rightarrow x = x_1$

$$N_1 = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

At node 2 $\Rightarrow x = x_2$

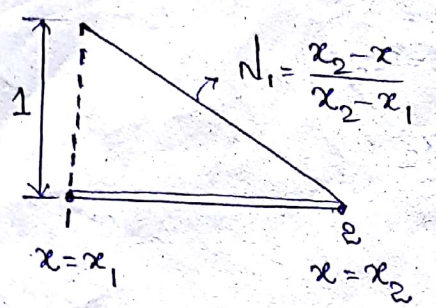
$$N_1 = \frac{x_2 - x_2}{x_2 - x_1} = 0.$$

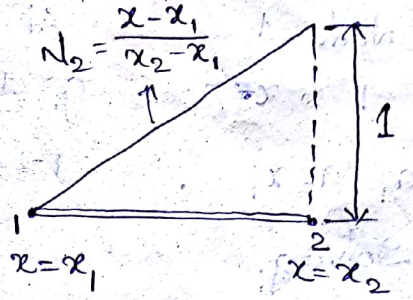
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Similarly $N_2 = \frac{x-x_1}{x_2-x_1}$

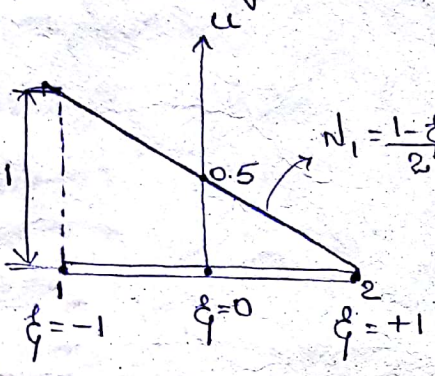
At Node 1 $\Rightarrow x = x_1$ $N_2 = 0$

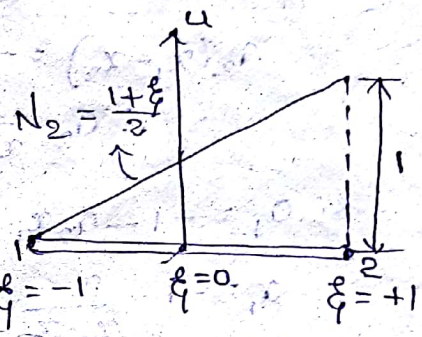
At Node 2 $\Rightarrow x = x_2$ $N_2 = 1$





Similarly, when $N_1 = \frac{1-\xi}{2}$ $N_2 = \frac{1+\xi}{2}$





(2). Completeness:

1. Always the sum of the (Nodal) shape functions is Equal to unity.

$$N_1 + N_2 = 1$$

Check:

$$\frac{x_2-x}{x_2-x_1} + \frac{x-x_1}{x_2-x_1} = \frac{x_2-x+x-x_1}{x_2-x_1} = \frac{x_2-x_1}{x_2-x_1}$$

2, $N_1 u_1 + N_2 u_2 = u$

& $N_1 x_1 + N_2 x_2 = x$ for all 'x'

Check

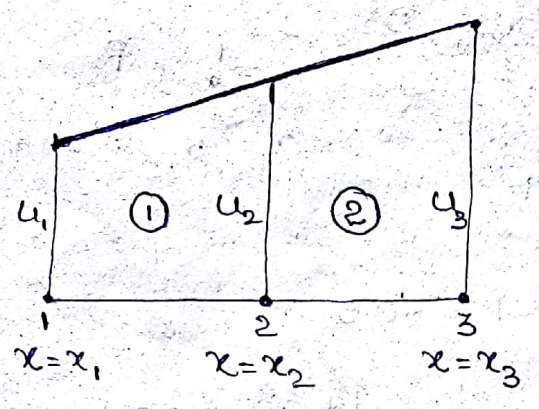
$$\frac{x_2-x}{x_2-x_1} \cdot x_1 + \frac{x-x_1}{x_2-x_1} \cdot x_2 = \frac{x_2x_1 - xx_1 + xx_2 - x_1x_2}{x_2-x_1}$$

$$N_1x_1 + N_2x_2 = \frac{x(-x_1+x_2)}{x_2-x_1} = x.$$

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(3) Compatibility:

The shape function should be a continuous function across the element boundary.



$$u^1 = \frac{x_2-x}{x_2-x_1}\cdot u_1 + \frac{x-x_1}{x_2-x_1}\cdot u_2 \quad \& \quad u^2 = \frac{x_3-x}{x_3-x_2}u_2 + \frac{x-x_2}{x_3-x_2}\cdot u_3$$

Check:-

Let $x=x_2$

$$u^1 = \frac{x_2-x_2}{x_2-x_1}\cdot u_1 + \frac{x_2-x_1}{x_2-x_1}u_2 = u_2$$

$$u^2 = \frac{x_3-x_2}{x_3-x_2}u_2 + \frac{x_2-x_2}{x_3-x_2}u_3 = u_2$$

(4) If the Nodal displacement is a Linear variation then the element displacement must be a linear variation.

(5) The sum of the derivatives of shape function must be zero.

$$\frac{\partial N_1}{\partial \xi} + \frac{\partial N_2}{\partial \xi} = 0 \quad \text{or}$$

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial x} = 0$$

Check:-

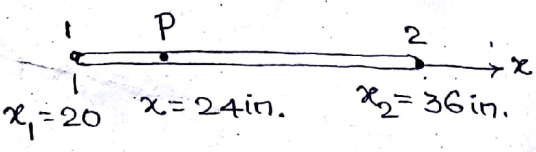
$$\frac{\partial}{\partial \xi}\left(\frac{1-\xi}{2}\right) + \frac{\partial}{\partial \xi}\left(\frac{1+\xi}{2}\right) = -\frac{\partial \xi}{\partial \xi} + \frac{\partial \xi}{\partial \xi} = -1+1 = 0.$$

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Problem :-

cont:

1. Consider the following fig: (a) Evaluate ξ, N_1, & N_2 at point p. & (b) If $q_1 = 0.003$ in & $q_2 = -0.005$ in, determine the value of the displacement q at point 'P'



Sol:- W.K.T

$$\xi = \frac{2(x-x_1)}{x_2-x_1} - 1$$

$$\xi = \frac{2}{16}(24-20) - 1 = -0.5$$

& $N_1 = \frac{1-\xi}{2} = 0.75$ & $N_2 = \frac{1+\xi}{2} = 0.25$.

(b) W.K.T

$$u = u_1N_1 + u_2N_2$$
$$= 0.75(0.003) + 0.25(-0.005)$$
$$= 0.001 \text{ in.}$$

* Strain Displacement Matrix :

$$\epsilon = \frac{du}{dx}$$

$$\epsilon = \frac{du}{d\xi} \times \frac{d\xi}{dx}$$

W.K.T $$\xi = \frac{2(x-x_1)}{x_2-x_1} - 1$$

$$\frac{d\xi}{dx} = \frac{2}{x_2-x_1}$$

We know the Isoparametric formulation $u = q_1N_1 + q_2N_2$

$$u = \frac{1-\xi}{2}q_1 + \frac{1+\xi}{2}q_2$$

$$\frac{du}{d\xi} = -\frac{q_1}{2} + \frac{q_2}{2} = -\frac{q_1+q_2}{2}$$

$$\epsilon = \frac{du}{d\xi} \times \frac{d\xi}{dx} = \frac{-q_1+q_2}{2} \times \frac{2}{(x_2-x_1)}$$

$$= \frac{1}{(x_2-x_1)}(-q_1+q_2)$$

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$$\epsilon = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\epsilon = Bq$$

$$\therefore B = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

Where B = Strain-displacement Matrix.

& Stress $= E\epsilon$

$$\sigma = EBq.$$

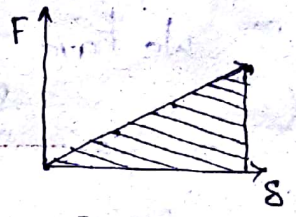
Potential Energy Approach :

We know that

π = Strain Energy ± Work potential.
(U) (W_p).

We have $U = \frac{1}{2} F \times \delta$

$= \frac{1}{2} \times \sigma \times A \times \epsilon l.$



$= \frac{1}{2} \sigma . \epsilon . V.$ $\quad \because \sigma = \frac{F}{A}, \epsilon = \frac{\delta}{l}, V = A \times l.$

$U = \frac{1}{2} \times$ Stress $\times$ Strain $\times$ Volume.

it can be written as

$$U = \frac{1}{2} \sigma^T \epsilon . V.$$

In Integration form.

$$U = \int_V \frac{1}{2} \sigma^T \epsilon \, dv.$$

Let work done = Force x Displacement.

W_p = Body force + Traction force + Point load force.

$$= -\int_V u^T f . dv - \int_S u^T T . ds - \sum_{i=1}^{n} P_i u_i$$

$$\therefore \pi = \int_V \frac{1}{2} \sigma^T \epsilon . dv - \int_V u^T f . dv - \int_S u^T T . ds - \sum_{i=1}^{n} P_i u_i^T$$

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Since, the Domain or Continuum has been discretized into finite elements, the Expression of π becomes,

$$\pi = \sum_e \frac{1}{2}\int_e \sigma^T \epsilon A.dx - \sum_e \int u^T f A.dx - \sum_e \int_e u^T T.dx - \sum_i Q_i^T P_i.$$

→ Element Stiffness Matrix :

Consider the Strain Energy

$$U_e = \frac{1}{2}\int_e \sigma^T \epsilon A.dx.$$

$$\sigma = EBq \quad , \quad \epsilon = Bq$$

$$U_e = \frac{1}{2}\int_e q^T.B^T EBq A.dx.$$

$$U_e = \frac{1}{2} q^T \int_e [B^T.EBA.dx] q.$$

We have

$$\frac{dx}{d\xi} = \frac{x_2 - x_1}{2} \text{ or } \frac{d\xi}{dx} = \frac{2}{x_2 - x_1}$$

$$dx = \frac{x_2 - x_1}{2} \times d\xi.$$

$$dx = \frac{l_e}{2} d\xi \text{ where } -1 \le \xi \le 1, \& \ l_e = x_2 - x_1$$

$$\therefore U_e = \frac{1}{2} q^T B.AB^T E_e \int_e \left[\frac{x_2 - x_1}{2} d\xi\right] q$$

$$= \frac{1}{2} q^T \left[A_e \frac{l_e}{2} E_e B^T.B \int_{-1}^{+1} d\xi\right] q$$

We have, $\int_{-1}^{+1} d\xi = \xi\Big]_{-1}^{+1} = 1+1 = 2$, & We know that

$B = \frac{1}{l_e}[-1 \ \ 1]$; & $B^T = \frac{1}{l_e}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$= \frac{1}{2} q^T A_e \frac{l_e}{2} E_e \frac{1}{l_e^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1 \ \ +1] \times 2 \times q.$$

$$= \frac{1}{2} q^T \frac{A_e E_e}{l_e} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \times q.$$

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$$U_e = \frac{1}{2} q^T K_e q$$

Where k_e = Element Stiffness Matrix

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

→ Force Terms :

Consider the body force terms. $\int_e u^T f . A . dx$

$$\int_e u^T f . A_e dx = \int_e f . A_e \left(N_1 q_1 + N_2 q_2\right)^T . dx$$

$$= A_e f \int_e \left[\begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \right]^T . dx .$$

$$= A_e f . q^T \begin{bmatrix} \int N_1 dx \\ \int N_2 dx \end{bmatrix}$$

Let $dx = \frac{l_e}{2} d\xi$, $-1 \le \xi \le +1$, we have $N_1 = \frac{1-\xi}{2}$ & $N_2 = \frac{1+\xi}{2}$

$$\int N_1 dx = \frac{l_e}{2} \int_{-1}^{+1} N_1 d\xi = \frac{l_e}{2} \int_{-1}^{+1} \left(\frac{1-\xi}{2}\right) d\xi = \frac{l_e}{2}$$

$$\int N_2 dx = \frac{l_e}{2} \int_{-1}^{+1} N_2 d\xi = \frac{l_e}{2} \int_{-1}^{+1} \left(\frac{1+\xi}{2}\right) d\xi = \frac{l_e}{2}$$

$$\therefore \int_e u^T f . A_e dx = q^T . A_e f \frac{l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= q^T . f^e$$

Where f^e = Element Body force vector

$$= \frac{A_e f l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

→ Traction force :- Here $ds = dx$. Perimeter.

Consider the element traction force term

$$\int u^T . T . dx = \int_e \left(N_1 q_1 + N_2 q_2\right)^T . T . dx .$$

$$= T q^T \begin{Bmatrix} \int N_1 dx \\ \int N_2 dx \end{Bmatrix}$$

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Where, $\int_e N_1 dx = \int_e N_1 dx = l_e/2$.

$$\int_e u^T T \, dx = q^T T^e$$

T^e = Element Traction Force.

$$= \frac{Tl_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix}.$$

We finded the element Matrices k^e, f^e, T^e.

The total Potential Energy can be Written as.

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F.$$

Where K = Global stiffness Matrix

Q = Global displacement Vector.

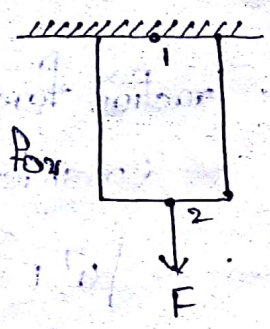
F = Global Load vector.

* Treatment of Boundary Conditions :

For any system under analysis, the information about the Nature of Loading in different locations, Nature of nodal displacements and other fields variables such as the values of temperatures at different Nodes etc are considered as boundary conditions. The boundary conditions are of two types.

1. Geometric or Essential boundary Conditions.
2. Natural or Non-Essential boundary Conditions.

* In the figure, the fixed Node at 1 is the Essential boundary Condition.
* The load at the free end is an Example for Non-Essential boundary Condition.



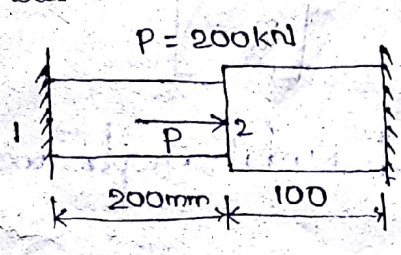
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Two Approaches are discussed for handling specified displacement boundary conditions. They are

1. Elimination Approach.
2. Penalty Approach.

* Problems on Elimination Approach :

1. Determine the Nodal displacements, stresses & strains induced in the bar as shown.



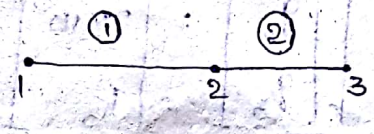
$A_1 = 1000mm^2$ $A_2 = 2000mm^2$

$E_1 = 200GPa$ $E_2 = 83GPas$

$= 200 \times 10^9 N/m^2$ $= 0.83 \times 10^5 N/mm^2$

$= 2 \times 10^5 N/mm^2$

Sol :-(i) Discrilization of structure :



Element	Nodes	
①	1	2
②	2	3

(ii) & (iii) Assuming an Appropriate solution & obtaining element stiffness Matrix & load vector.

$$k^e = \frac{A_e E_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k^1 = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1000 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore k^1 = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 1 & 2 \\ 10^6 & -10^6 \\ -10^6 & 10^6 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$k^2 = \frac{2000 \times 0.83 \times 10^5}{100} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{matrix} \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 2 & 3 \\ 1.6 \times 10^6 & -1.6 \times 10^6 \\ -1.6 \times 10^6 & 1.6 \times 10^6 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

And From the figure $F_2 = 200kN = 200 \times 10^3 N$. &

R_1 & R_3 are the Reaction forces.

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(iv) Global Stiffness Matrix & global force vectors:

$$K = K^1 + K^2 = \begin{bmatrix} 10^6 & -10^6 & 0 \\ -10^6 & 10^6 + 1.6\times10^6 & -1.6\times10^6 \\ 0 & -1.6\times10^6 & +1.6\times10^6 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$F = \begin{bmatrix} F_1 + T_1 + P_1 \\ F_2 + T_2 + P_2 \\ F_3 + T_3 + P_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 2\times10^5 \\ R_3 \end{bmatrix}$$

(v) Solving the Unknowns: (a) To find Nodal Displacements.

$$KQ = F$$

$$\begin{bmatrix} 10^6 & -10^6 & 0 \\ -10^6 & 10^6 + 1.6\times10^6 & -1.6\times10^6 \\ 0 & -1.6\times10^6 & 1.6\times10^6 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 2\times10^5 \\ R_3 \end{bmatrix} \longrightarrow (A)$$

Here, the Nodes at 1 and 3 are fixed, therefore the Nodal displacements at 1 and 3 are zero.

$\therefore q_1 = q_3 = 0.$

By using the principle of Elimination Approach; Eliminate 1 row 1 column, & 2 row, 2nd column in the Matrix.

$$\begin{bmatrix} 10^6 & -10^6 & 0 \\ -10^6 & 10^6 + 1.6\times10^6 & -1.6\times10^6 \\ 0 & -1.6\times10^6 & 1.6\times10^6 \end{bmatrix} \begin{bmatrix} 0 \\ q_2 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ 2\times10^5 \\ R_3 \end{bmatrix}$$

$$\therefore (10^6 + 1.6\times10^6) q_2 = 2\times10^5$$

$$(2.6\times10^6) q_2 = 2\times10^5$$

$$q_2 = 0.076$$

$$q_1 = 0$$

$$q_3 = 0.$$

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(b) To find Reaction forces :

The Point load applied at Node 2 is resisted by the reaction forces developed at Nodes 1 and 3 (i.e at fixed ends) in order to Maintain the Equilibrium Condition of the system.

From, the global finite Element Matrix (A)

We can write the Equations for the reaction forces at Nodes 1 and 3, such as

$$-10^6 q_2 = R_1 \Rightarrow -10^6(0.076) = F_1 \quad | \because q_2 = 0.076.$$

$$\therefore R_1 = -76 \times 10^3 \text{ N.} = -76 \text{ KN}$$

& $$-1.6 \times 10^6 q_2 = R_3$$

$$R_3 = -121.6 \times 10^3 \text{ N.} = -121.6 \text{ KN.}$$

The negative sign to F_1 and F_3 is due to their directions of action which are opposite to the Applied force F_2.

(vi) Stresses & Strains :

ϵ_1 for Element 1 $= \frac{u_2 - u_1}{l_1} = \frac{0.076}{200} = 3.84 \times 10^{-4}$

ϵ_2 for Element 2 $= \frac{q_3 - q_2}{l_2} = \frac{0 - 0.076}{100} = -7.69 \times 10^{-4}$

σ_1 for Element 1 $= E_1 \epsilon_1 = 2 \times 10^5 \times 3.84 \times 10^{-4}$

$= 76.8 \text{ N/mm}^2$

σ_2 for Element 2 $= E_2 \epsilon_2 = 0.83 \times 10^5 \times -7.69 \times 10^{-4}$

$= -63.8 \text{ N/mm}^2$

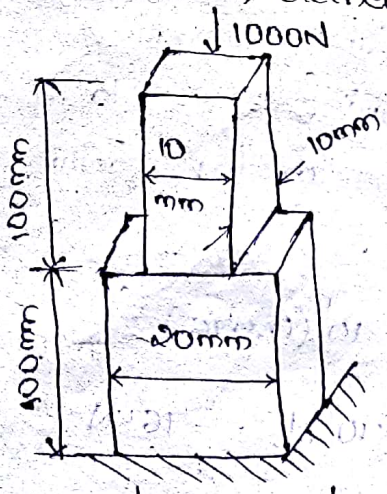
(or)

The strains & stresses can be also calculated by using the formula.

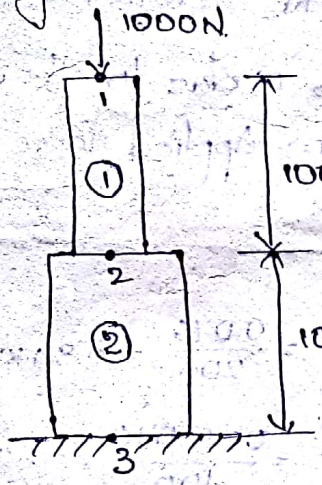
$$\epsilon = Bq, \quad \sigma = EBq, \quad \& \quad B = \frac{1}{l_e}[-1 \quad 1]$$

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2. For the column made of Mild steel as shown in Fig. Find (a) Nodal displacement, (b) Reaction force at the supports (c) Stresses & Strains in Elements. Assume $E = 2\times10^5\ N/mm^2$



Sol :- The given column can be redrawn as spring element model consisting of 2 elements & 3 Nodes.



from figure

$A_1 = 10\times10 = 100\ mm^2$

$A_2 = 20\times10 = 200\ mm^2$

$E_1 = E_2 = 2\times10^5\ N/mm^2$

$l_1 = l_2 = 100mm$.

(a) Nodal displacements :

Consider the Element Stiffness Matrix

$$k^1 = \frac{A_1E}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{100\times2\times10^5}{100}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 2\times10^5\ \begin{matrix}1 & 2\\ \end{matrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{matrix}1\\ 2\end{matrix}$$

$$k^2 = \frac{A_2E}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{200\times2\times10^5}{100}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 2\times10^5\ \begin{matrix}2 & 3\\ \end{matrix}\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}\begin{matrix}2\\ 3.\end{matrix}$$

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Global stiffness Matrix $K = k^1 + k^2$

$$K = 2\times10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

The finite element eq $[K]Q = \{F\}$

$$2\times10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ F_3 \end{bmatrix}$$

From the boundary conditions.

$q_3 = 0$, $F_1 = 1000$, $F_2 = 0$, so eliminate 3rd row & Column.

$$2\times10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ -F_3 \end{bmatrix}$$

$$2\times10^5 \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

$$2\times10^5 (q_1 - q_2) = 1000$$

$$2\times10^5 (-q_1 + 3q_2) = 0$$

$$\Rightarrow -q_1 + 3q_2 = 0$$

By solving the Above equations, we get

$q_1 = 75\times10^{-4}$ mm & $q_2 = 25\times10^{-4}$ mm.

(b) Reaction forces :-

The Reaction force is at Node F_3.

$$2\times10^5 (-2q_2) = F_3$$

$$F_3 = -1000N \text{ (Acting Upwards)}.$$

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(c) Stress & Strains :-

For element 1

$$\epsilon_1 = Bq = \frac{1}{l_1}[-1 \quad 1]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= \frac{1}{100}[-1 \quad 1]\begin{Bmatrix} -75\times10^{-4} \\ 25\times10^{-4} \end{Bmatrix}$$

$$= -50\times10^{-6}$$

$$\sigma_1 = EBq = 2\times10^5 \times -50\times10^{-6} = -10\ N/mm^2$$

For element 2

$$\epsilon_2 = Bq = \frac{1}{l_2}[-1 \quad 1]\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

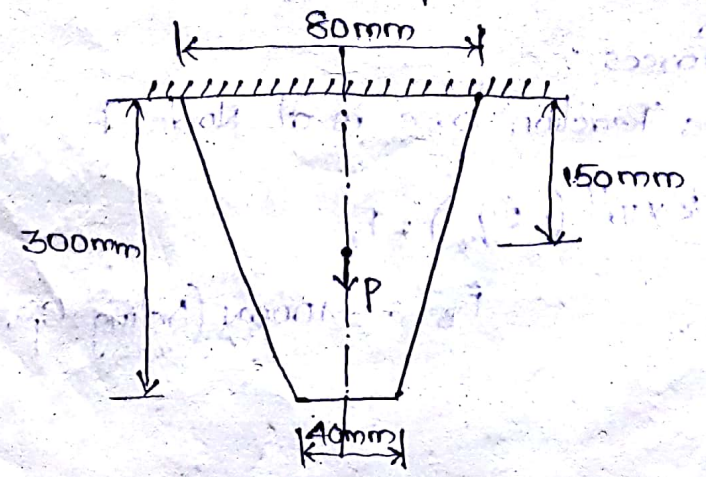
$$= \frac{1}{100}[-1 \quad 1]\begin{Bmatrix} 25\times10^{-4} \\ 0 \end{Bmatrix}$$

$$= -25\times10^{-6}$$

$$\sigma_2 = EBq = 2\times10^5(-25\times10^{-6}) = -5\ N/mm^2$$

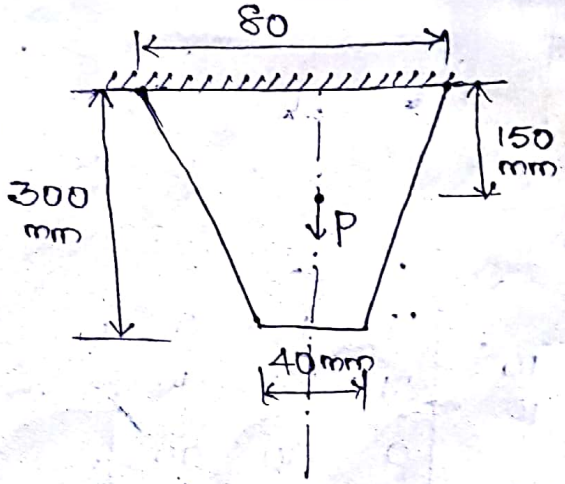
The Negative sign indicates Compressive stresses.

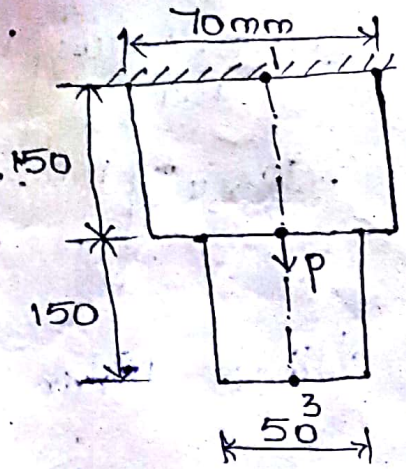
6. For a tapered bar of Uniform thickness $t = 10mm$ as shown Find the displacements at the nodes by deforming into two elements. The bar has a Mass density $\rho = 7800\ kg/m^3$ Young's Modulus $E = 2\times10^5\ MN/m^2$. In addition to self weight, the bar is subjected to a point load $P = 1KN$ at its centre. Also determine the reaction force at the Support.



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Sol





$P = 1000\,N$

$\rho = 7800\,kg/m^3$

$E = 2\times10^5\,MN/m^2$

$t = 10mm$

Now for the tapered bar

The area of Node 1 = Width x thickness

$= 80\times10 = 800mm^2$

The area at Node 3 = Width & Thickness

$= 40\times10 = 400mm^2$

The Area at Node 2 $= \frac{\text{Area at Node 1} + \text{Area at Node 3}}{2}$

$= \frac{800+400}{2}$

$= 600mm^2$

for the stepped bar

$A_1 = \frac{(\text{Area at Node 1} + \text{Area at Node 2})}{2}$

$= \frac{800+600}{2} = 700mm^2$ = 70mm width x 10mm thickness

$A_2 = \frac{\text{Area at Node 2} + \text{Area at Node 3}}{2}$

$= \frac{600+400}{2} = 500mm^2$ = 50mm Width x 10mm thickness

(a) To find Nodal Displacements :-

for element 1

$$K^1 = \frac{A_1E_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{700\times2\times10^5}{150}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{14}{15}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

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$$k^2 = \frac{500 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{10}{15} \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global stiffness Matrix $K = \frac{10^6}{15}$

$$\begin{matrix} & 1 & 2 & 3 \\ \end{matrix}$$
$$\begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

To find the force vectors, Since the bar is having a self weight, it is assumed that the body force due to self weight of the element is equally distributed in its two nodes.

We know that, for Element 1

$$f_1 = \frac{A_1 l_1 P_1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \frac{700 \times 150 \times 7800 \times 10^{-9} \times 9.81 \times 10^3}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_2^1$$

$g = 9.81 \text{ m/sec}^2$
$= 9.81 \times 10^3 \text{ mm/sec}^2$

$\because f = \rho g v$
ρ is at unit volume
$f = \rho g$

$$f_1 = 4.017 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 4.017 \end{Bmatrix}_2^1 \times 10^3$$

For Element 2

$$f_2 = \frac{A_2 l_2 P_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

3.

$$= \frac{500 \times 150 \times 7800 \times 10^{-9} \times 9.81 \times 10^3}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_3^2$$

$$f_2 = 2.869 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2.869 \\ 2.869 \end{Bmatrix}_2^1 \times 10^3$$

$\therefore$ Global force vector

$$F = \begin{Bmatrix} f_1 + T_1 + P_1 \\ f_2 + T_2 + P_2 \\ f_3 + T_3 + P_3 \end{Bmatrix}_3^2 = \begin{Bmatrix} 4.017 \times 10^3 \\ (4.017 + 2.869 + 1000) \times 10^3 \\ 2.869 \times 10^3 \end{Bmatrix}_3^2$$

$$= \begin{Bmatrix} 4.017 \\ 1886 \\ 2.869 \end{Bmatrix}$$

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(12)
(13)

$\therefore \{K\}Q = \{F\}$.

$$\frac{10^6}{15}\begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Here, At Node 1 $q_1 = 0$ & F_1 is the reaction force and also having a body force

$\therefore F_1 = f_1 + R_1$

Since $q_1 = 0$, Eliminate 1st row & 1st column.

$$\frac{10^6}{15}\begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 4.017 + R_1 \\ 7886.00 \\ 2.869 \end{bmatrix}$$

$$\frac{10^6}{15}\begin{bmatrix} 24 & -10 \\ -10 & 10 \end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 7886.00 \\ 2.869 \end{bmatrix}$$

$$\frac{10^6}{15}\left[24q_2 - 10q_3\right] = 7886.00$$

$$\frac{10^6}{15}\left[-10q_2 + 10q_3\right] = 2.869$$

By solving, We get

$q_2 =$, $q_3 =$

To find the Reaction forces

$$\frac{10^6}{15}\left[-14q_2\right] = 4.017 + R_1$$

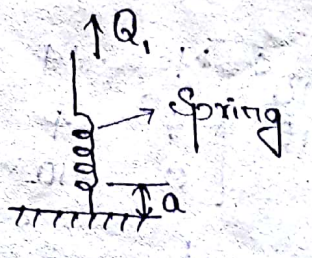
$$R_1 = \frac{10^6}{15}\left[-14q_2\right] - 4.017 \times 10^3$$

$\therefore R_1 =$

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Penalty Approach :

* $$\begin{bmatrix} K_{11}+C & K_{12} & K_{13} & \cdots & K_{1N} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2N} \\ \vdots & & & & \\ K_{N1} & K_{N2} & \cdots & \cdots & K_{NN} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_3 \end{bmatrix}$$



$$= \begin{bmatrix} F_1 + Ca_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix}$$

Where $C = \text{Max} |K_{ij}| \times 10^4$, $1 \le i \le N$, $1 \le j \le N$

Here, the value of C is added at the fixed Node places.

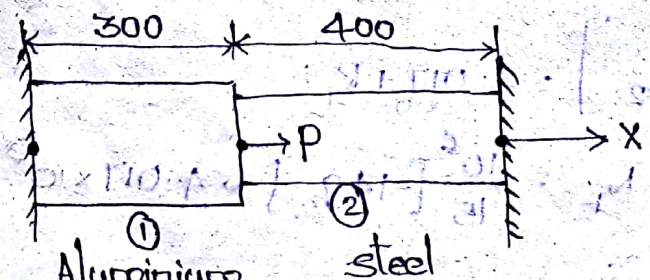
Problem :-

* Consider the bar shown as Fig. An axial load $P = 200 \times 10^3 N$ is applied as shown. Using the penalty Approach for handling boundary conditions, do the following.

(a) Determine the Nodal displacements.

(b) Determine the stress in each Material.

(c) Determine the ~~stress~~ Reation forces.



$A_1 = 2400 mm^2$ $A_2 = 600 mm^2$

$E_1 = 70 \times 10^9 N/m^2$ $E_2 = 200 \times 10^9 N/m^2$

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Sol:- (a) Element stiffness Matrix

(14)

$$k^1 = \frac{A_1E_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{2400\times70\times10^3}{300}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 0.56\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 10^6\begin{bmatrix}0.56 & -0.56\\ -0.56 & 0.56\end{bmatrix}\begin{matrix}1\\2\end{matrix}$$

$$k^2 = \frac{600\times200\times10^3}{400}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 0.3\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 10^6\begin{bmatrix}0.3 & -0.3\\ -0.3 & 0.3\end{bmatrix}\begin{matrix}2\\3\end{matrix}$$

Global stiffness Matrix $k = k^1 + k^2$

$$k = 10^6\begin{bmatrix}0.56 & -0.56 & 0\\ -0.56 & 0.86 & -0.3\\ 0 & -0.3 & 0.3\end{bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

Global load vector

$$F = \begin{bmatrix}0 & 200\times10^3 & 0\end{bmatrix}^T$$

Here, as 1 and 3 are fixed, when using the penalty approach therefore, a large no. C is added to the first and third diagonal element of 'k' and the value 'Ca' is added to the 1 & 3 values of 'F'

$$\therefore q_1 = 0 \Rightarrow a_1 = 0, \quad q_2 = 0 \Rightarrow a_2 = 0.$$

$$\& \quad C = \text{Max}|k_{ij}|\times10^4 = [0.86\times10^6]\times10^4.$$

$$C = 8600\times10^6$$

$$k = 10^6\begin{bmatrix}8600.56 & -0.56 & 0\\ -0.56 & 0.86 & -0.3\\ 0 & -0.3 & 8600.3\end{bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

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& Modified Load Vector $F = \begin{bmatrix} F_1 + CQ_1 \\ F_2 \\ F_3 + CQ_3 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{bmatrix}$$

$$\therefore 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 8600.3 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{bmatrix}$$

By solving the above Matrix, We get

$Q_1 = 15.14 \times 10^{-6}$ $Q_2 = 0.232$ & $Q_3 = 8.11 \times 10^{-6}$

(b) the element stresses.

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 15.14 \times 10^{-6} \\ 0.232 \end{Bmatrix}$$

$$= 54.27 \text{ MPas} \quad | \quad 1 \text{ MPas} = 1 \times 10^6 \text{ N/m}^2 = 1 \text{ N/mm}^2$$

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.232 \\ 8.11 \times 10^{-6} \end{Bmatrix}$$

$$= -116.29 \text{ N/mm}^2 \text{ or MPa}$$

(c) Reaction forces

$$R_1 = -CQ_1 = -\left[0.86 \times 10^6 \times 10^4\right] \times 15.1432 \times 10^{-6}$$

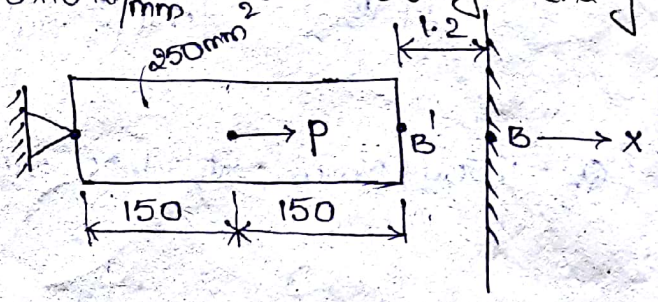
$$= -130.23 \times 10^3 \text{ N}.$$

$$R_2 = -CQ_3 = -\left[0.86 \times 10^{10}\right] \times 8.11 \times 10^{-6}$$

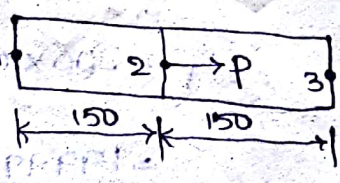
$$= -69.77 \times 10^3 \text{ N}.$$

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2. In fig, a load $P = 60\times10^3$ N is applied as shown. Determine the displacement field, stress & Support Reactions in the body Take $E = 20\times10^3$ N/mm^2 & solve using Penalty Approach.



Sol: (a) Element Stiffness Matrix



$$k^1 = \frac{250\times20\times10^3}{150}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{10^5}{3}\begin{matrix}\\1\\2\end{matrix}\begin{matrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{matrix}\begin{matrix}1 & 2\end{matrix}$$

$$k^2 = \frac{250\times20\times10^3}{150}\begin{bmatrix}-1 & -1\\ +1 & 1\end{bmatrix}$$

$$= \frac{10^5}{3}\begin{matrix}\\2\\3\end{matrix}\begin{matrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{matrix}\begin{matrix}2 & 3\end{matrix}$$

∴ Global Stiffness Matrix

$$k = \frac{10^5}{3}\begin{bmatrix}-1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

& Global load vector

$$F = \begin{bmatrix}0, 60\times10^3, 0\end{bmatrix}$$

Again, Using the penalty Approach, the value of c is to be added to the 1st & 3rd diagonal elements.

$$C = \frac{2}{3}\times10^5\times10^4 = 20000\times\frac{10^5}{3}$$

& $q_1 = 0$ & $q_3 = 1.2$ & $Q_1 = 0$.

And the value $cq_3 = C\times1.2$ is added to the 3rd component F.

$$\frac{10^5}{3}\begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60\times10^3 \\ 0+\left(\frac{2}{3}\times10^9\times1.2\right) \end{Bmatrix}$$

By solving

$Q_1 = 7.49\times10^{-5}$, $Q_2 = 1.50045mm$, $Q_3 = 1.200015mm$.

(b) Stresses

$$\sigma_1 = 200\times10^3\times\frac{1}{150}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix} 7.499\times10^{-5} \\ 1.50045 \end{Bmatrix}$$

$$= 199.99\ N/mm^2$$

$$\sigma_2 = 200\times10^3\times\frac{1}{150}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix} 1.50045 \\ 1.200015 \end{Bmatrix}$$

$$= -40.004\ N/mm^2$$

(c) Reation forces.

$$R_1 = -C\times7.499\times10^{-5} = -\frac{2}{3}\times10^9\times7.499\times10^{-5}$$

$$= -49.99\times10^3 N.$$

$$R_2 = -C\times(Q_3-q_3) = -\frac{2}{3}\times10^9\times(1200015-1.2)$$

$$= -9999.99\ N.$$

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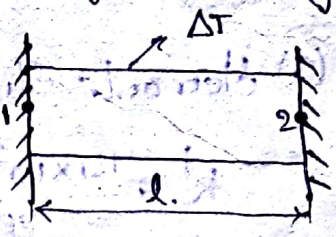
* Temperature Effects :

When a Machine member is loaded at room temperature its dimensions change and the corresponding stresses will be induced in the Machine member. Sometimes, without the application of external loading the member will expand or contract due to change of temperature. If the member is allowed to expand or contract freely, even for the change of temperature & corresponding change in dimensions, no stresses will be induced in the member.

If the ends of the elements are restricted, during the rise or fall of temperature, a stress called 'thermal stress' is induced in the member which may be a Tensile or Compressive in Nature.

The strain due to change of temperature is known as thermal strain.

Let consider a bar whose ends are fixed & named as nodes 1 & 2 as shown in figure, undergo a change of temperature ΔT.



ΔT = Rise in Temp.(°C).

α = Coefficient of Thermal Expansion (mm/mm°C).

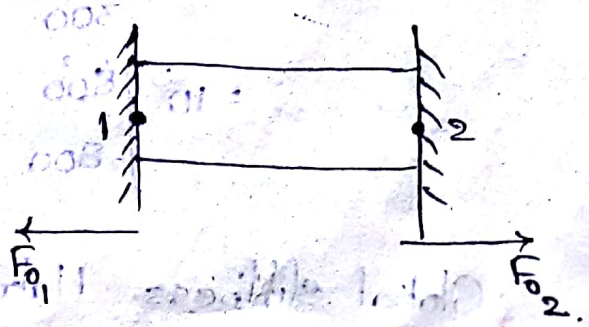
then,

Thermal strain, $\epsilon_0 = \alpha \Delta T$.

Thermal stress, $\sigma_0 = E\alpha\Delta T$.

Thermal Force, $F_0 = AE\alpha\Delta T$.

$\therefore [F_0] = \begin{Bmatrix} F_{01} \\ F_{02} \end{Bmatrix} = EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$



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If the ~~Meber~~ Member, is subjected to both external loading and change in temperature, the resultant stress induced in the bar

$$\sigma_r = \sigma - \sigma_0 = E\epsilon - E\epsilon_0$$
$$= EBq - E\alpha\Delta T$$

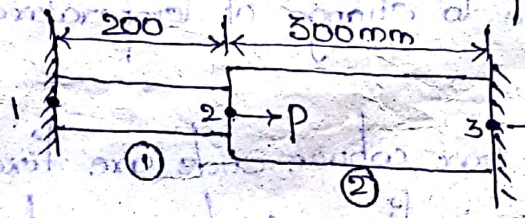
$$\sigma_x = \frac{E}{x_2 - x_1}[-1 \quad 1]\,q - E\alpha\Delta T$$

(or)

$$F = \sum_e (f^e + T^e + \theta^e) + P$$

Problems :-

1. An Axial load $P = 300 \times 10^3$ N is applied at 20°c to the rod as shown in fig. the temperature is then raised to 60°c
   (a) Assemble K & F Matrices.
   (b) Determine the Nodal Disp. & element stresses.



Aluminium ① Steel ②

$E_1 = 70 \times 10^9$ N/m² $E_2 = 200 \times 10^9$ N/m²

$A_1 = 900$ mm² $A_2 = 1200$ mm²

$\alpha_1 = 23 \times 10^{-6}$/°C $\alpha_2 = 11.7 \times 10^{-6}$/°C

Sol:-

(a) Element Stiffness Matrices.

$$k^1 = \frac{70 \times 10^3 \times 900}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 315 & -315 \\ -315 & 315 \end{bmatrix}_2^1$$

$$k^2 = \frac{200 \times 10^3 \times 1200}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 800 & -800 \\ -800 & 800 \end{bmatrix}_3^2$$

Global stiffness Matrix $K = 10^3 \begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix}$ N/mm.

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Now, to find the Global load vector 'F', both temperature and point load effects have been considered.

$$\theta' = EA\alpha\Delta T\begin{bmatrix}-1\\1\end{bmatrix} \quad \& \quad \Delta T = 60-20 = 40^{\circ}C$$

$$= 70\times10^{3}\times900\times23\times10^{-6}\times40\begin{Bmatrix}-1\\1\end{Bmatrix}\begin{matrix}1\\2\end{matrix} \text{ N.}$$

$$\theta' = 10^{3}\begin{Bmatrix}-57.96\\57.96\end{Bmatrix}\begin{matrix}1\\2\end{matrix}$$

$$\theta^{2} = 200\times10^{3}\times1200\times11.7\times10^{-6}\times40\begin{Bmatrix}-1\\1\end{Bmatrix}\begin{matrix}2\\3\end{matrix}$$

$$= 10^{3}\begin{Bmatrix}-112.32\\112.32\end{Bmatrix}\begin{matrix}2\\3\end{matrix}$$

$$\therefore F = 10^{3}\begin{Bmatrix}-57.96\\+57.96-112.32+300\\112.32\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

$$F = 10^{3}\begin{Bmatrix}-57.96\\245.64\\112.32\end{Bmatrix} \text{ N}$$

(b)

Consider $KQ = F$.

$$10^{3}\begin{bmatrix}315 & -315 & 0\\-315 & 1115 & -800\\0 & -800 & 800\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix} = \begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix} = \begin{Bmatrix}-57.96\times10^{3}\\245.64\times10^{3}\\112.32\times10^{3}\end{Bmatrix}$$

By Using the Elimination Approach, $q_1 = 0$ & $q_3 = 0$, So Eliminate 1st & 3rd rows & Columns.

$$10^{3}[1115]q_2 = 10^{3}\times245.64$$

$$q_2 = 0.22\text{mm}$$

$$\therefore Q = [0, 0.22, 0]^{T}\text{mm.}$$

Stresses: $\sigma_1 = \frac{E}{l_e}[-1\ \ 1]\begin{Bmatrix}q_1\\q_2\end{Bmatrix} - E\alpha\Delta T$

$$= \frac{70\times10^{3}}{900}[-1\ \ 1]\begin{Bmatrix}0\\0.22\end{Bmatrix} - 70\times10^{3}\times23\times10^{-6}\times40.$$

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$= 12.60 \text{ MPa}$

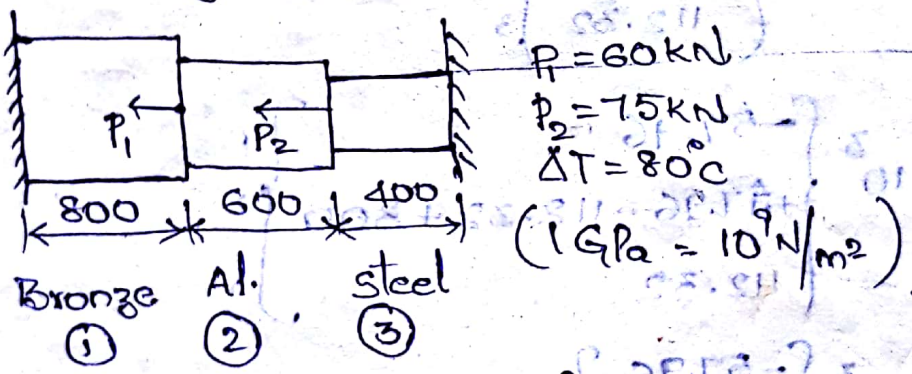
8)

$$\sigma_2 = \frac{200\times10^3}{300}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0.22\\0\end{Bmatrix} - 200\times10^3\times11.7\times10^{-6}\times40.$$

$$= -240.27 \text{ MPa}.$$

2. Assignment :

The structure is subjected to an increase in temp, $\Delta T = 80$. Determine the displacements, stresses, & support reactions. Solve this problem using elimination method for handling B.C.



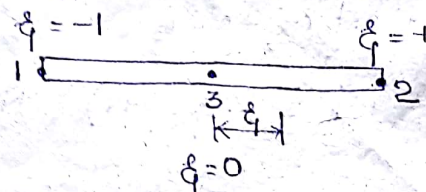
$(1 \text{ GPa} = 10^9 \text{ N/m}^2)$

A	2400mm^2	1200mm^2	600mm^2
E	83GPa	70GPa	200GPa
α	$18.9\times10^{-6}/°C$	$23\times10^{-6}/°C$	$11.7\times10^{-6}/°C$.

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Quadratic shape functions :

Consider typical three nodes element as shown.



Consider the polynomial equation

$$u = a_1 + a_2\xi + a_3\xi^2$$

At Node 1 : $u = u_1$ & $\xi = -1$

$$u_1 = a_1 - a_2 + a_3 \longrightarrow (i)$$

At Node 2 : $u = u_2$ & $\xi = +1$

$$u_2 = a_1 + a_2 + a_3 \longrightarrow (ii)$$

At Node 3 : $u = u_3$ & $\xi = 0$

$$u_3 = a_1 \longrightarrow (iii)$$

Solving (i) & (ii)

$$u_1 = a_1 - a_2 + a_3$$
$$u_2 = a_1 + a_2 + a_3$$
$$u_1 - u_2 = -2a_2$$
$$a_2 = \frac{u_2 - u_1}{2}$$

& $a_1 = u_3$

$$\therefore a_3 = u_1 - a_1 + a_2$$
$$= u_1 - u_3 + \frac{u_2 - u_1}{2}$$
$$= \frac{2u_1 - 2u_3 + u_2 - u_1}{2}$$
$$a_3 = \frac{u_1 + u_2 - 2u_3}{2}$$

$$u = u_3 + \frac{u_2 - u_1}{2}\xi + \frac{u_1 + u_2 - 2u_3}{2}\xi^2$$
$$= -\frac{u_1}{2}\xi + \frac{u_1}{2}\xi^2 + \frac{u_2}{2}\xi + \frac{u_2}{2}\xi^2 + u_3 - u_3\xi^2$$
$$= \frac{u_1}{2}(\xi^2 - \xi) + \frac{u_2}{2}(\xi^2 + \xi) + u_3(1 - \xi^2)$$

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$$u = \frac{1}{2}u_1\xi(\xi-1) + \frac{1}{2}\xi(\xi+1)u_2 + (1-\xi^2)u_3.$$

Comparing with $u = N_1u_1 + N_2u_2 + N_3u_3$

$$N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = \frac{1}{2}\xi(\xi+1), \quad N_3 = (1-\xi^2)$$
$$= (1+\xi)(1-\xi)$$

Another procedure :-

For example since $N_1 = 0$ at $\xi = 0$ and $N_1 = 0$ at $\xi = 1$ and $N_1 = 1$ at $\xi = -1$,

$$N_1 = C_1\xi(\xi-1)$$
$$1 = C_1(-1)(-2)$$
$$C_1 = 1/2$$
$$\therefore N_1 = \frac{1}{2}\xi(\xi-1)$$

& $N_2 = C_2(1+\xi)(\xi-1)$

$N_2 = 1$ & $\xi = 0, +1$

$$1 = C_2(2)(1)$$
$$C_2 = 1/2$$
$$\Rightarrow N_2 = \frac{1}{2}\xi(1+\xi)$$

Similarly, $N_3 = C_3(\xi+1)(\xi-1)$

$N_3 = 1$ and $\xi = 0$.

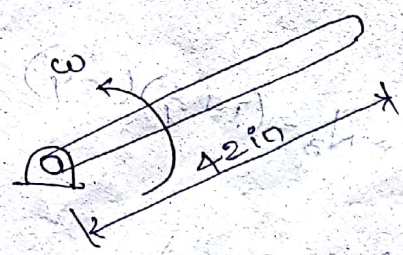
$$1 = C_3(1)(-1)$$
$$C_3 = -1$$
$$N_3 = -1(\xi+1)(\xi-1)$$
$$= -1(\xi^2-1)$$
$$\therefore N_3 = (1-\xi^2).$$

$$\therefore N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = \frac{1}{2}\xi(1+\xi), \quad N_3 = (1-\xi^2)$$

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Problem:-

1: Consider the Rod as shown in fig, which is rotating at constant angular velocity $\omega = 30$ rad/sec. Determine the axial stress distribution in the rod, using 2 quadratic elements. Consider only the centrifugal force Ignore bending of the Rod.

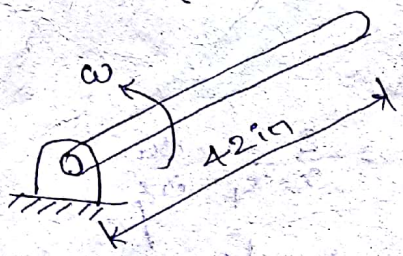


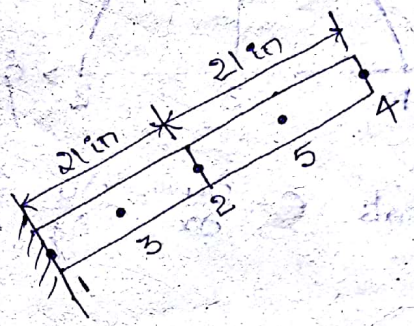
$A = 0.6 in^2$

$E = 10^7$ psi

Weight Density, $\rho = 0.2836$ lb/in^3.

Sol:-





Element	Nodes		
①	1	2	3
②	2	4	5

We know that for quadratic element

$$k^1 = \frac{E_e A_e}{3l_e}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} = \frac{10^7 \times 0.6}{3 \times 21}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

$$k^2 = \frac{10^7 \times 0.6}{3 \times 21}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{matrix}2\\4\\5\end{matrix}$$

$$K = \frac{10^7 \times 0.6}{3 \times 21}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{matrix}1\\2\\3\end{matrix} + \frac{10^7 \times 0.6}{3 \times 21}\begin{bmatrix} 7 & 1 & -8 \\ 7 & 1 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{matrix}2\\4\\5\end{matrix}$$

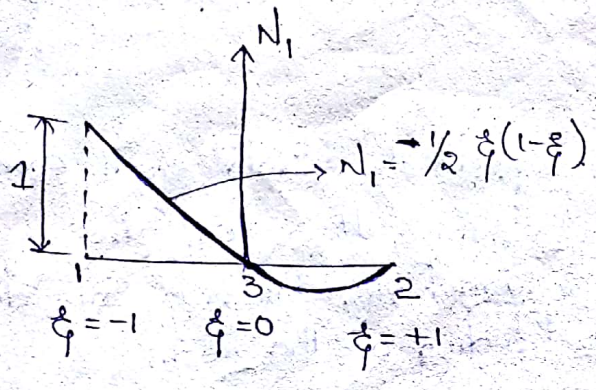
$$K = \frac{10^7 \times 0.6}{3 \times 21}\begin{bmatrix} 7 & 1 & -8 & 0 & 0 \\ 1 & 14 & -8 & 1 & -8 \\ -8 & -8 & 16 & 0 & 0 \\ 0 & 1 & 0 & 7 & -8 \\ 0 & -8 & 0 & -8 & 16 \end{bmatrix}\begin{matrix}1\\2\\3\\4\\5\end{matrix}$$

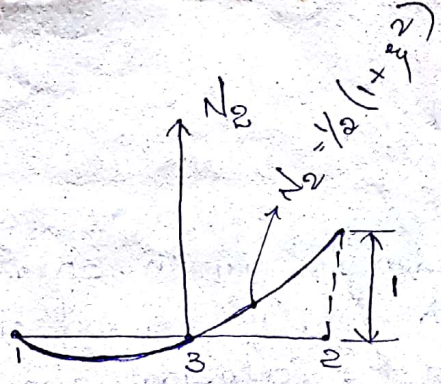
We know that, the centrifugal force

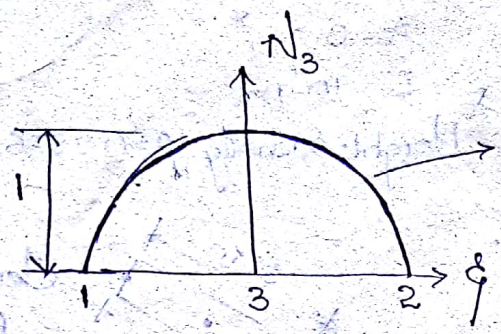
$$f = \frac{\rho r \omega^2}{g} \text{ lb/in}^3$$

$\rho = 0.2836$ lb/in^3, $g = 32.2$ ft/s^2

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We know that $\epsilon = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{2}{x_2 - x_1} \cdot \frac{\partial u}{\partial \xi}$

$$= \frac{2}{x_2 - x_1}\left[\frac{\partial}{\partial \xi}\left[N_1 q_1 + N_2 q_2 + N_3 q_3\right]\right]$$

$$\epsilon = \frac{2}{(x_2 - x_1)}\begin{bmatrix}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\end{bmatrix}\begin{bmatrix}q_1 \\ q_2 \\ q_3\end{bmatrix}$$

Here $N_1 = \frac{1}{2}(\xi^2 - 1)$ & $-\frac{1}{2}(1-\xi^2)$

$$N_2 = \frac{1}{2}(1+\xi^2)$$

$$N_3 = (1+\xi)(1-\xi) = (1 - \xi + \xi - \xi^2)$$

$$\epsilon = \frac{2}{(x_2 - x_1)}\begin{bmatrix}-\frac{(1-2\xi)}{2} & \frac{(1+2\xi)}{2} & -2\xi\end{bmatrix} q$$

$$\therefore B = \frac{2}{x_2 - x_1}\begin{bmatrix}-\frac{(1-2\xi)}{2} & \frac{1+2\xi}{2} & -2\xi\end{bmatrix}$$

Consider the Potential Energy eq :-

$$\Pi = \sum \frac{1}{2}\int \sigma^T \epsilon A \cdot dx - \sum_e \int_e u^T f A dx - \sum \int_a u^T T \cdot dx - \sum Q_i P_i$$

By solving, we get

$$k_e = \frac{E_e A_e}{3 l_e}\begin{bmatrix}7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16\end{bmatrix}, \quad f^e = \frac{A_e l_e f}{2}\int_{-1}^{+1} N^T d\xi = A_e l_e f \begin{Bmatrix}1/6 \\ 1/6 \\ 2/3\end{Bmatrix}$$

$$\& \quad T^e = l_e T \begin{Bmatrix}1/6 \\ 1/6 \\ 2/3\end{Bmatrix}$$

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Element ①

$$f_1 = \frac{0.2836 \times 10.5 \times 30^2}{32.2 \times 12} = 6.94 \quad \Big| \because r_1 = \frac{21}{2} = 10.5$$

Element ②

$$f_2 = \frac{0.2836 \times 31.5 \times 30^2}{32.2 \times 12} = 20.81. \qquad r_2 = 21 + \frac{21}{2} = 21 + 10.5 = 31.5$$

We know that the element body force

$$f^1 = AL f_1 \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} = 0.6 \times 21 \times 6.94 \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} = \begin{Bmatrix} 14.57 \\ 14.57 \\ 58.29 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$f^2 = 0.6 \times 21 \times 20.81 \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \begin{matrix} 2 \\ 5 \\ 4 \end{matrix} = \begin{Bmatrix} 43.7 \\ 43.7 \\ 174.80 \end{Bmatrix} \begin{matrix} 2 \\ 5 \\ 4 \end{matrix}$$

$$\therefore \text{Total force, } f = \begin{Bmatrix} 14.57 \\ 58.27 \\ 58.29 \\ 174.80 \\ 43.7 \end{Bmatrix}$$

Consider the finite element equation

$$KQ = F$$

$$\frac{10^7 \times 0.6}{3 \times 21} \begin{bmatrix} 7 & 1 & -8 & 0 & 0 \\ 1 & 14 & -8 & 1 & -8 \\ -8 & -8 & 16 & 0 & 0 \\ 0 & 1 & 0 & 1 & -8 \\ 0 & -8 & 0 & -8 & 16 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} 14.57 \\ 58.27 \\ 58.29 \\ 174.80 \\ 43.7 \end{bmatrix}$$

Since Node 1 is fixed, by using the elimination approach, Eliminate 1st rows & 1st column then we get

$$\frac{10^7 \times 0.6}{3 \times 21} \begin{bmatrix} 14 & -8 & 1 & -8 \\ -8 & 16 & 0 & 0 \\ 1 & 0 & 1 & -8 \\ -8 & 0 & -8 & 16 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} 14.57 \\ 58.27 \\ 58.29 \\ 174.80 \\ 43.7 \end{bmatrix}$$

By solving we get

q_2, q_3, q_4 & q_5 Values.

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And we have stress

$$\sigma = EBq$$

where $B = \frac{2}{l_e}\left[-\frac{(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right]$

Element ①

$$\sigma_1 = 10^7 \times \frac{2}{21}\left[-\frac{(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Nod stress at Node 1, $\xi = -1$

$$\sigma_{1/1} = 10^7 \times \frac{2}{21}\begin{bmatrix} 1.5 & 0.5 & 2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Stress at Node 2, $\xi = +1$

$$\sigma_{1/2} = 10^7 \times \frac{2}{21}\begin{bmatrix} 0 & 3/2 & -2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

& Stress at Node 3, $\xi = 0$.

$$\sigma_{1/3} = 10^7 \times \frac{2}{21}\begin{bmatrix} -0.5 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

By Substituting the values of q_1, q_2, q_3 we get the stress at Node 1, 2, & 3.

& Element ②

$$\sigma_2 = 10^7 \times \frac{2}{21}\left[-\frac{(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right]\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

At Node 2, $\xi = -1$

$$\sigma_{2/2} = 10^7 \times \frac{2}{21}\begin{bmatrix} -1.5 & 0.5 & 2 \end{bmatrix}\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

At Node 4, $\xi = 1$

$$\sigma_{2/4} = 10^7 \times \frac{2}{21}\begin{bmatrix} 0 & 3/2 & -2 \end{bmatrix}\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

At Node 5, $\xi = 0$

$$\sigma_{2/5} = 10^7 \times \frac{2}{21}\begin{bmatrix} -0.5 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

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