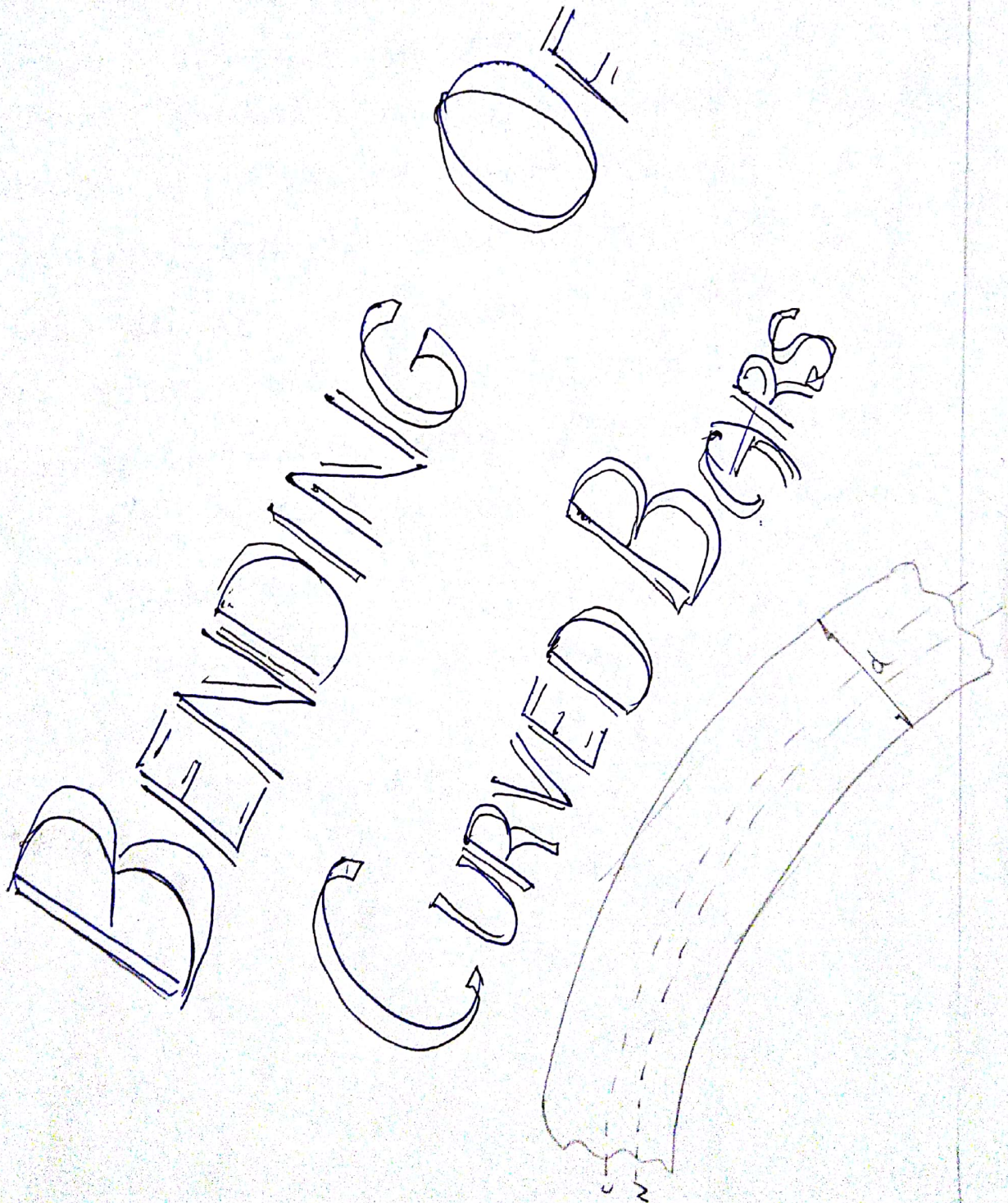
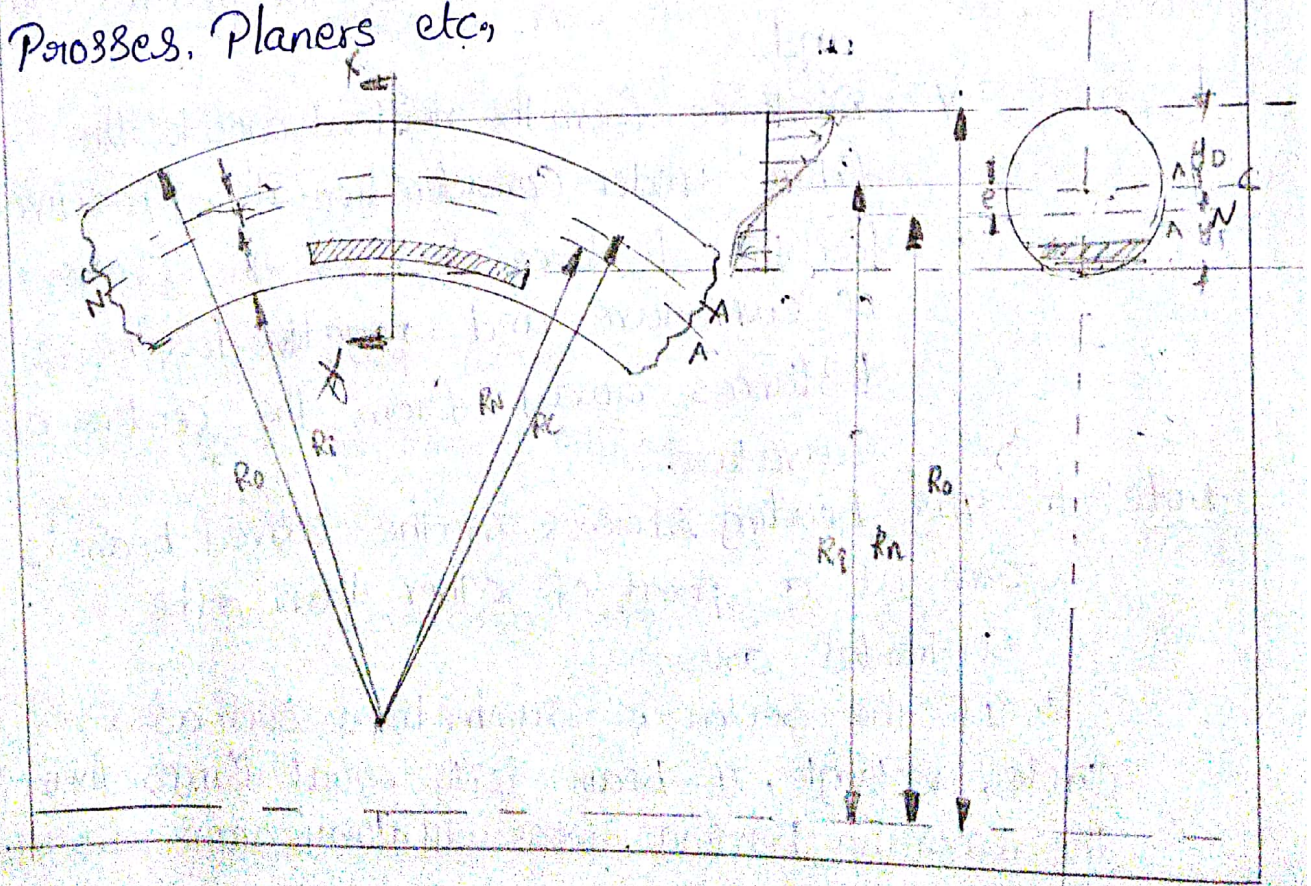


UNIT-III
CURVED BEAMS



Bending Stress in Curved Beams:

We have seen in the Previous article that for the straight beams, the neutral axis of the section coincides with its centroidal axis. The stress distribution in the beam is linear. But in case of curved beams, the neutral axis of the cross-section is shifted towards the centre of curvature of the beam causing a non-linear (hyperbolic) distribution of stress, as shown in Fig. 5.8. It may be noted that the neutral axis lies between the Centroidal axis and Centre of Curvature and always occurs within the curved beams. The application of curved beam principle is used to crane hooks, chain links and frames of Punches, Presses, Planers etc.



Consider a curved beam subjected to a bending moment M , as shown in Fig. 5.8. In finding the bending stress in curved beams, the same assumptions are used as for straight beams. The general expression for the bending stress (σ_b) in a curved beam at any fibre at the distance y from the neutral axis is given by

$$\sigma_b = \frac{M}{A \cdot e} \left(\frac{y}{R_n - y} \right)$$

Where M = Bending moment acting the given section about the centroidal axis.

A = Area of Crossing Section

e = Distance from the centroidal axis to the ^{neutral} ~~centroidal~~ axis = $R - R_n$

R = Radius of Curvature of the Centroidal axis.

R_n = Radius of Curvature of the neutral axis and

y = Distance from the neutral axis to the fibre under consideration. It is positive for the distance towards the Centre of Curvature and negative for the distances away from the Centre of Curvature.

Note:- 1. The bending stress in the curved beam is zero at a point other than at the Centroidal axis

2. If the section is symmetrical such as a circle, rectangle, I-beam with equal flanges, then the maximum bending stress will always occur fibre

3. If the section is symmetrical, then the maximum bending stress may occur at the either the inside fibre or the outside fibre. The maximum bending stress at the inside fibre is given by

$$\sigma_{bi} = \frac{M \cdot y_i}{A \cdot e \cdot R_i}$$

Where

y_i = Distance from the neutral axis to the inside fibre = $R_o - R_i$ and

R_o = Radius of curvature of the outside fibre

It may be noted that the bending stress at the inside fibre is tensile while the bending stress at the outside fibre is compressive.

4. If the section has an axial load in addition to bending, then the axial or direct stress (σ_d) must be added algebraically to the bending stress, in order to obtain the resultant stress on the section. In other words

Resultant stress, $\sigma = \sigma_d \pm \sigma_b$

The following table shows the values of R_o and R_i for various commonly used cross-section in the curved beam.

Introduction:-

In chapter 5 bending equation $\left(\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}\right)$ was derived assuming the beam to be initially straight (besides other fundamental assumption) before the application of bending moment. However, machine members and structures subjected to bending are not always straight, as in the case of car's hooks, chain links etc., before a bending moment is applied to them. The simple flexure formula may be used for curved beams of which radius of curvature is more than five times the depth of beam. The simple bending formula, however is not applicable for deeply curved beams where the neutral and centroidal axes do not coincide. To deal with such cases Winkler-Bach Theory is used.

STRESSES IN CURVED BARS

(Winkler-Bach Theory)

Fig 20.1 shows a bar ABCD initially in its unstrained state. Let ABCD be the strained position of the bar.

Let, R = Radius of curvature of the centroidal axis H G₁.

y : Distance of fibre EF from the centroidal layer H G₁.

R_1 : Radius of curvature of HG_1'

y_1 : Distance between the EF' and HG_1' after straining

M' : Uniform bending moment applied to the beam (assumed positive when bending to increase curvature)

θ : Original angle subtended by centroidal axis HG at its centre of curvature O , and

θ' : Angle subtended by HG_1' at its centre of curvature

The following assumptions are made in this analysis

1. Plan section (transverse) remains plan during bending
2. The material obeys Hooke's law (limit of proportionality is not exceeded)
3. Radial strain is negligible
4. The fibres are free to expand or contract without any constraining effect from the adjacent fibres.

For finding the strain and stress normal to the section

Consider the fibre EF at a distance y from the Centroidal axis.

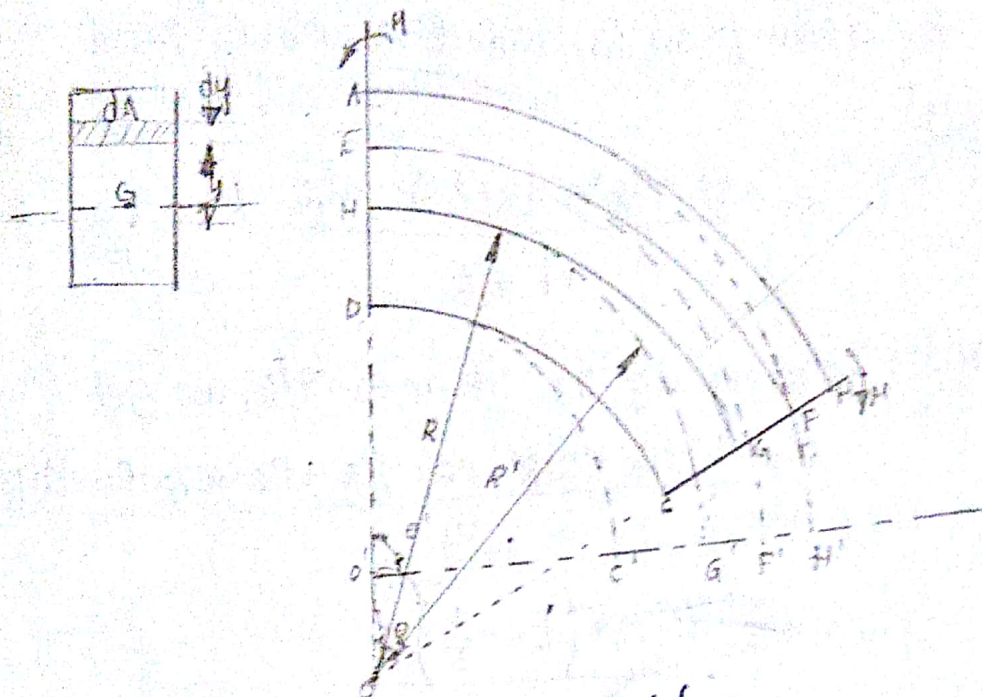


Fig 20.1 Bending Curved bar

Let be the stress in the strained layer EF' under the bending moment M and e is strain in the same layer

$$\text{Strain } e = \frac{EF' - EF}{EF} = \frac{(R' + y')\theta' - (R + y)\theta}{(R + y)\theta} \quad \text{or, } e = \frac{R' + y'}{R + y} \frac{\theta'}{\theta} - 1$$

Also e_0 = strain in the Centroidal layer i.e., when $y = 0$

$$= \frac{R'\theta'}{R\theta} - 1 \quad \text{or} \quad 1 + e = \frac{R' + y'}{R + y} \frac{\theta'}{\theta}$$

$$\text{and} \quad 1 + e_0 = \frac{R'}{R} \cdot \frac{\theta'}{\theta}$$

Dividing equation (20.1) and (20.2) we get

$$\frac{1 + e}{1 + e_0} = \frac{R' + y'}{R + y} \cdot \frac{R}{R'} = \frac{1 + y'/R}{1 + y/R}$$

$$\text{or} \quad e = (1 + e_0) \left[\frac{1 + y'/R}{1 + y/R} \right] - 1$$

$$\text{or} \quad e = \frac{e_0 - y/R + y/R' + e_0 - y/R}{1 + y/R}$$

According to assumption (3), radius strain is zero

$$\therefore y = y'$$

Strain,

$$e = \frac{e_0 \cdot y/R' + y/R' + e_0 - y/R}{1 + y/R}$$

Adding and subtracting the term $e_0 \cdot y/R$, we get

$$e = \frac{e_0 \cdot y/R' + y/R' + e_0 - y/R + e_0 \cdot y/R - e_0 \cdot y/R}{1 + \frac{y}{R}}$$

or

$$e = \frac{(1+e_0) \left(\frac{1}{R'} - \frac{1}{R} \right) y}{1 + \frac{y}{R}}$$

From the Fig. 20.1 it's obvious that for the given bending moment the layers above the Centroidal layer are in tension and layer below the Centroidal layer in Compression

Stress, $\sigma = Ee = E \left[e_0 + \frac{(1+e_0) \left(\frac{1}{R'} - \frac{1}{R} \right) y}{\left(1 + \frac{y}{R} \right)} \right]$

(where, E = Young's modulus of the material)

Total force on the section $F = \int \sigma \cdot dA$

Considering a small strip of elementary area dA , at a distance of y from the centroidal layer HG, we have

$$F = E \int e_0 \cdot dA + E \int \frac{(1+e_0) \left(\frac{1}{R'} - \frac{1}{R} \right) y \cdot dA}{1 + y/R}$$

$$= E \int e_0 \cdot dA + E (1+e_0) \left(\frac{1}{R'} - \frac{1}{R} \right) \int \frac{y}{1 + y/R} dA$$

or

$$F = E e_0 \cdot A + E (1+e_0) \left(\frac{1}{R'} - \frac{1}{R} \right) \int \frac{y}{1 + y/R} dA$$

where A : area of Cross-section of the beam

The total resisting moment is given by

$$M = \int \sigma \cdot y \cdot dA = E \int e_0 \cdot y \cdot dA + E \int \frac{(1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)}{1+y/R} \cdot y^2$$

$$: E \cdot e_0 \times 0 = E (1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \int \frac{y^2}{1+y/R} \cdot dA \quad [\because \int y \cdot dA = 0]$$

$$\therefore M = E (1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \int \frac{y^2}{1+y/R} \cdot dA$$

$$\text{Let, } \int \frac{y^2}{1+y/R} \cdot dA = Ah^2$$

Where, h^2 = a constant for the Cross-section of the beam

$$\therefore M = E (1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) Ah^2$$

$$\begin{aligned} \text{Now, } \int \frac{y}{1+y/R} \cdot dA &= \int \left(y - \frac{y^2}{R+y} \right) dA = \int y \cdot dA - \int \frac{y^2}{R+y} \cdot dA \\ &= 0 - \frac{1}{R} \int \frac{y^2}{1+y/R} \cdot dA \end{aligned}$$

$$\therefore \int \frac{y}{1+y/R} \cdot dA = -\frac{1}{R} \int \frac{y^2}{1+y/R} \cdot dA = -\frac{1}{R} Ah^2$$

Hence eqn (20.5) becomes

$$F = E e_0 \cdot A - E (1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \frac{Ah^2}{R}$$

Since transversal plane section remains Plan during bending

$$F = 0$$

$$\text{or, } 0 = E e_0 \cdot A - E (1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \frac{Ah^2}{R}$$

$$\text{or } E e_0 \cdot A = E (1+e_0) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \frac{Ah^2}{R}$$

or.

$$E_0 = E (1 + e_0) \left(\frac{1}{R'} - \frac{1}{R} \right) \frac{Ah^2}{R}$$

or

$$\frac{e_0 R}{h^2} = (1 + e_0) \left(\frac{1}{R'} - \frac{1}{R} \right)$$

Substituting the value of $(1 + e_0) \left(\frac{1}{R'} - \frac{1}{R} \right)$ in eqn (20.7), we get

$$M = E \cdot \frac{e_0 R}{h^2} \cdot Ah^2 = e_0 E A R.$$

or

$$e_0 = \frac{M}{E A R}$$

Substituting the value of e_0 in eqn (20.4) we get

$$\delta = \frac{M}{A R} + E \times \left(\frac{y}{1 + y/R} \right) \times \frac{e_0 R}{h^2}$$

$$= \frac{M}{A R} + E \times \frac{y}{1 + y/R} \times \frac{R}{h^2} \times \frac{M}{E A R} \text{ Substing } e_0 = \frac{M}{E A R} \text{ again}$$

$$= \frac{M}{A R} + \frac{M R}{A R} \times \frac{R y}{1 + y/R} \times \frac{1}{h^2}$$

$$\delta = \frac{M}{A R} \left[1 + \frac{R}{h^2} \times \left(\frac{y}{R + y} \right) \right]$$

On the other side of HG, y will be negative and stress will be compressive

$$\therefore \delta = \frac{M}{A R} \left[1 - \frac{R}{h^2} \left(\frac{y}{R - y} \right) \right]$$

When the bending moment applied in such a manner that it tends to decrease the curvature than eqn. (20.12) will give compressive stress and eqn. (20.13) tensile stress

Position of neutral axis:-

At the neutral axis, $\delta = 0$

$$\therefore \frac{M}{A R} \left[1 + \frac{R^2}{h^2} \left(\frac{y}{R + y} \right) \right] = 0$$

or.

$$\frac{R^2}{h^2} \left(\frac{y}{R + y} \right) = -1$$

$$R^2 y = -h^2 (R + y) = -R h^2 - h^2 y$$

$$\text{or, } y(R^2 + h^2) = -R h^2$$

$$y = \frac{-R h^2}{R^2 + h^2}$$

Since the neutral axis is located below the centroidal axis.

VALUES OF h^2 FOR VARIOUS SECTIONS!

We know

$$h^2 = \frac{1}{A} \int \frac{y^2}{1+y/R} \cdot dA$$

$$= \frac{R}{A} \int \frac{y^2}{R+y} \cdot dA$$

$$= \frac{R}{A} \left[\int y dA - \int R dA + \int \frac{R^2 dA}{R+y} \right]$$

$$= \frac{R}{A} \left[0 - RA + \int \frac{R^2 + dA}{R+y} \right]$$

$$h^2 = \frac{R^3}{A} \int \frac{dA}{R+y} - R^2$$

Rectangular Section:-

Fig 20.2 shows the rectangular section with Centre of Curvature O lying on YY -axis. xx -axis is the centroidal bending axis. Consider an elementary strip of width B and depth dy at a distance y from the centroidal layer.

Area of strip, $dA = B \cdot dy$

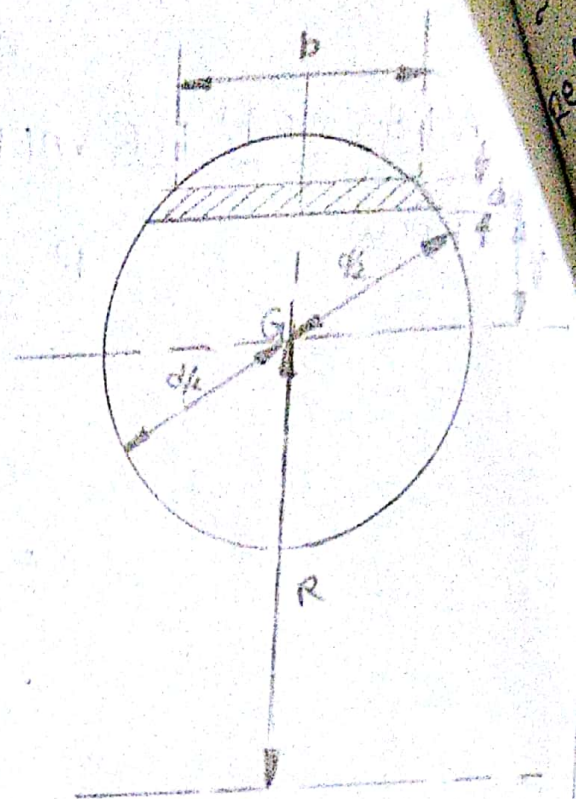
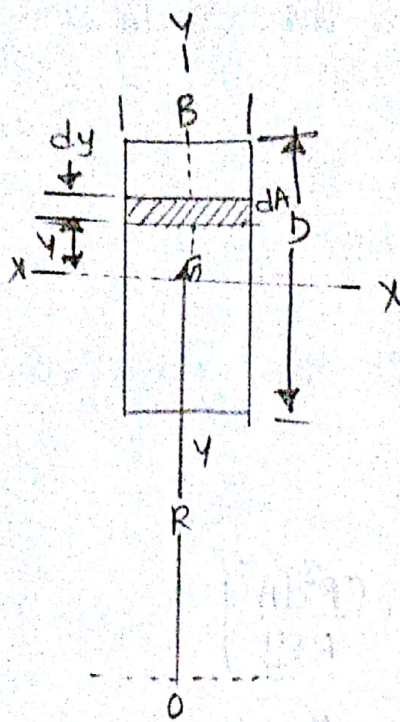
$$A = B \times D$$

$$h^2 = \frac{R^3}{B \times D} \int_{-D/2}^{+D/2} \frac{B \cdot dy}{R+y} - R^2$$

$$= \frac{R^3}{D} \left[\log_e (R+y) \right]_{-D/2}^{+D/2} - R^2$$

or.

$$h^2 = \frac{R^3}{D} \log_e \left(\frac{2R+D}{2R-D} \right) - R^2$$



Circular Section:-

Fig 20.3 shows the circular section of ~~distance of~~ diameter d of a curved bar of radius of curvature R , from the Centre of Curvature O up to the Centroid G of the section.

Area of Cross Section, $A = \frac{\pi}{4} d^2$

Consider a strip of width b and a depth dy at a distance y from the Centroidal layer as shown

$$b = 2\sqrt{\left(\frac{d}{2}\right)^2 - y^2} \cdot dy$$

Area of Strip $dA = bdy$

$$= 2\sqrt{\left(\frac{d}{2}\right)^2 - y^2} \cdot dy$$

$$h^2 = \frac{R^3}{A} \int_{-d/2}^{+d/2} \frac{2\sqrt{\left(\frac{d}{2}\right)^2 - y^2}}{R+y} \cdot dy - R^2$$

$$= \frac{8R^3}{\pi d^2} \int_{-d/2}^{+d/2} \frac{\frac{d}{4} - y^2}{R+y} \cdot dy - R^2$$

Expanding the integral by binomial expression and then integrating

$$h^2 = \frac{d^2}{16} + \frac{1}{128} \cdot \frac{d^4}{R^2} + \dots$$

Triangular Section:

Refer to Fig. 20.4

$$\text{Let } R+y=a$$

$$dy = da$$

Width of elementary strip

$$b' = \left(\frac{R_2 - a}{d} \right) b$$

Area of the elementary strip

$$dA = b' \cdot dy = b' \cdot da$$

$$\text{Now, } b^2 = \frac{R^3}{A} \int_{R_1}^{R_2} \frac{dA}{R+y} - R^2$$

$$= \frac{R^3}{A} \int_{R_1}^{R_2} \left(\frac{R_2 - a}{d} \right) b \cdot \frac{da}{a} - R^2$$

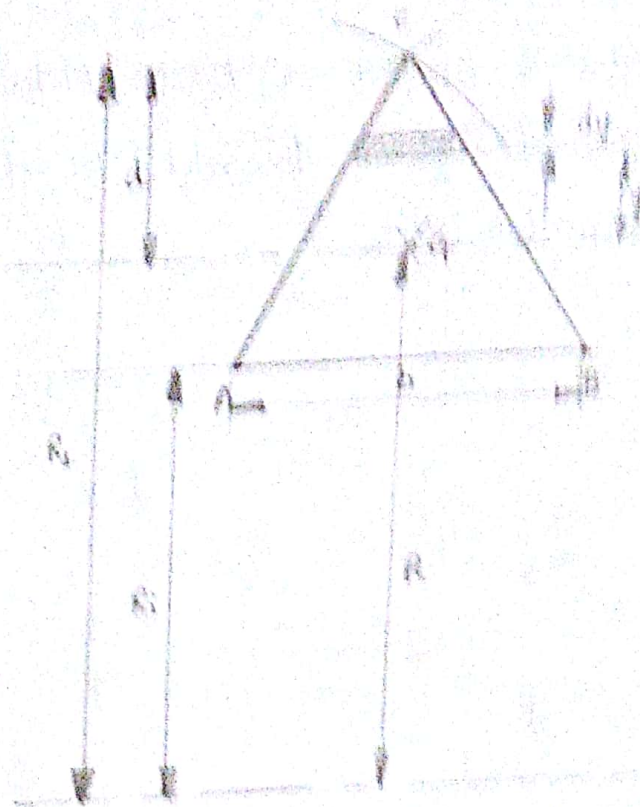
$$= \frac{R^3}{A} \left[\frac{R_2 b}{d} \log_e \frac{R_2}{R_1} - \frac{b}{d} (R_2 - R_1) \right] - R^2$$

$$= \frac{R^3}{A} \left[\left(d + \frac{2b}{3} \right) \frac{b}{d} \cdot \log_e \left(\frac{3R+2d}{3R-d} \right) - b \right] - R^2$$

$$\left[\begin{array}{l} \therefore R_2 = R + \frac{d}{3}, \text{ and} \\ R_1 = R - \frac{d}{3}, \text{ and} \\ R_2 - R_1 = d \end{array} \right]$$

$$= \frac{2R^3}{bd} \left[(3R+2d) \frac{b}{3d} \cdot \log_e \left(\frac{3R+2d}{3R-d} \right) - b \right] - R^2$$

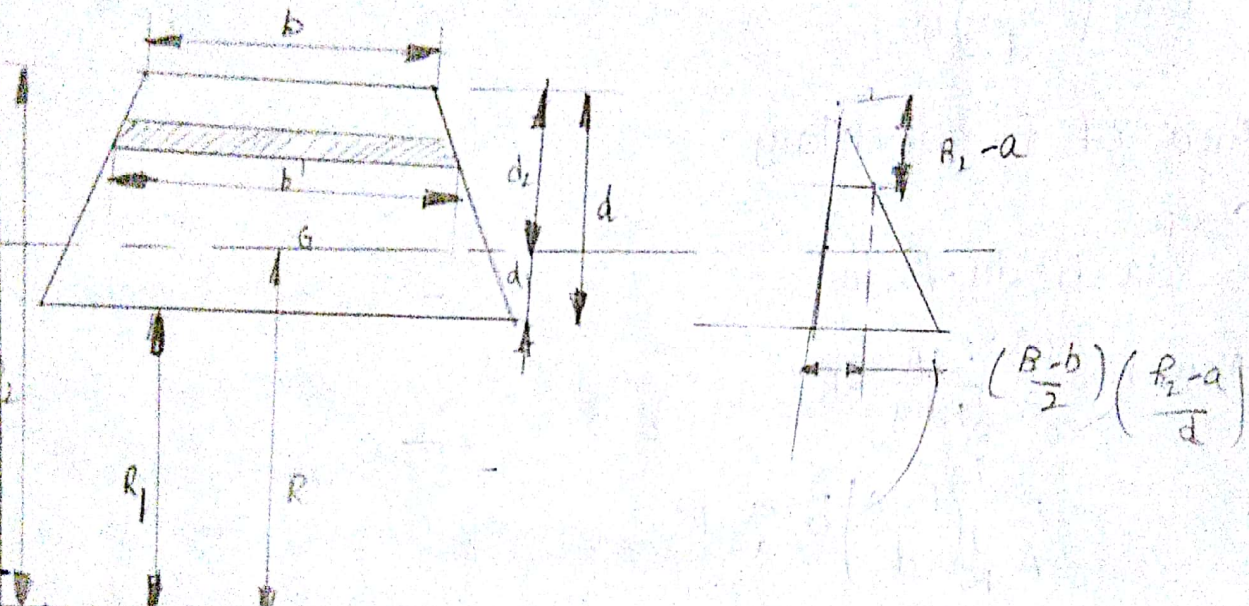
$$\text{or, } b^2 = \frac{2R^3}{d} \left[\left(\frac{3R+2d}{3d} \right) \log_e \left(\frac{3R+2d}{3R-d} \right) - 1 \right] - R^2$$



Triangular section

Trapezoidal Section:-

Fig 20.5 shows a trapezoidal section. Consider an elemental strip of width b' and dy at distance y from the Centroidal axis



$$\text{Let, } R+y = a$$

$$dy = da$$

$$b' = b + \left(\frac{B-b}{d+d_2} \right) (R_2 - a)$$

Area of the strip

$$dA = b' dy = \left[b + \left(\frac{B-b}{d_1+d_2} \right) (R_2 - a) \right] da$$

$$I^2 = \frac{R^3}{A} \int_{R_1}^{R_2} \left[b + \left(\frac{B-b}{d_1+d_2} \right) (R_2 - a) \right] \frac{da}{a} - R^2$$

$$= \frac{R^3}{A} \left[\int_{R_1}^{R_2} \frac{b da}{a} + \left(\frac{B-b}{d_1+d_2} \right) \int_{R_1}^{R_2} \left(\frac{R_2 - a}{a} \right) da \right] - R^2$$

$$= \frac{R^3}{A} \left[b \log_e \frac{R_2}{R_1} + \left(\frac{B-b}{d_1+d_2} \right) \left[R_2 \log_e a - a \right]_{R_1}^{R_2} \right] - R^2$$

$$= \frac{R^3}{A} \left[b \cdot \log_e \frac{R_2}{R_1} + \left(\frac{B-b}{d_1+d_2} \right) \left\{ R_2 \log_e \frac{R_2}{R_1} - (R_2-R_1) \right\} \right] - R^2$$

$$= \frac{R^3}{A} \left[b \cdot \log_e \left(\frac{R+d_2}{R-d_1} \right) + \left(\frac{B-b}{d_1+d_2} \right) (R+d_2) \log_e \left(\frac{R+d_2}{R-d_1} \right) - (B-b) \right] - R^2$$

$$= \frac{R^3}{A} \left[b \cdot \log_e \left(\frac{R+d_2}{R-d_1} \right) + \left(\frac{B-b}{d} \right) (R+d_2) \log_e \left(\frac{R+d_2}{R-d_1} \right) - (B-b) \right] - R^2$$

Where

$$A = \left(\frac{B+b}{2} \right) d$$

$$d_1 = \frac{d}{3} \left(\frac{B+2b}{B+b} \right)$$

$$d_2 = d - d_1$$

T-Section

Fig 20.6 shows a T-section

Let, $R+y=a$

$$dy = da$$

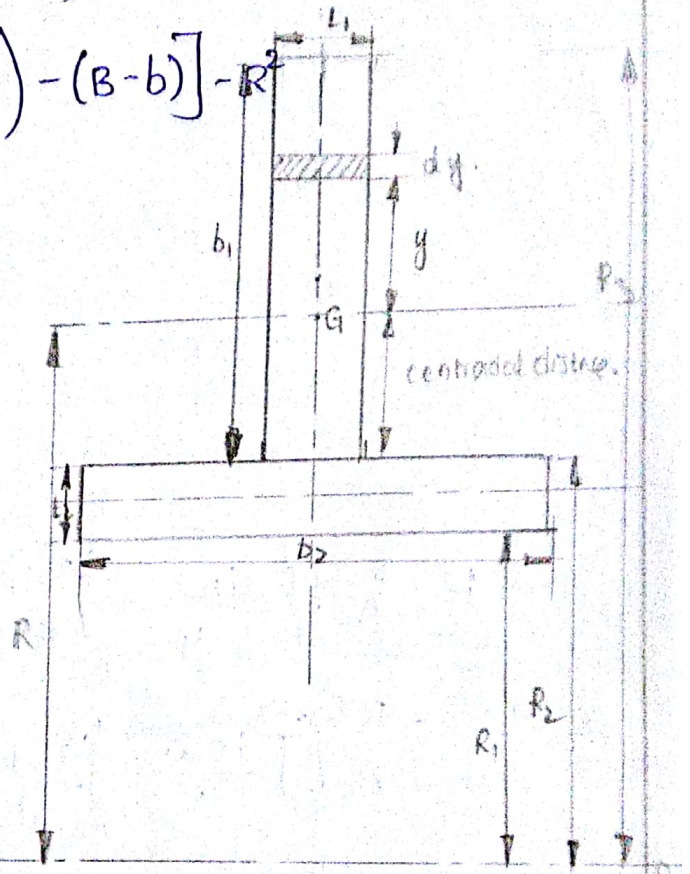
$$h^2 = \frac{R^3}{A} \left[\int_{R_1}^{R_2} \frac{dA}{R+y} + \int_{R_2}^{R_3} \frac{dA}{R+y} \right] - R^2$$

$$= \frac{R^3}{A} \left[\int_{R_1}^{R_2} \frac{b_2 da}{a} + \int_{R_2}^{R_3} \frac{t_1 da}{a} \right] - R^2$$

$$= \frac{R^3}{A} \left[b_2 \log_e \frac{R_2}{R_1} + t_1 \log_e \frac{R_3}{R_2} \right] - R^2$$

$$\text{i.e. } h^2 = \frac{R^3}{A} \left[b_2 \log_e \frac{R_2}{R_1} + t_1 \log_e \frac{R_3}{R_2} \right] - R^2$$

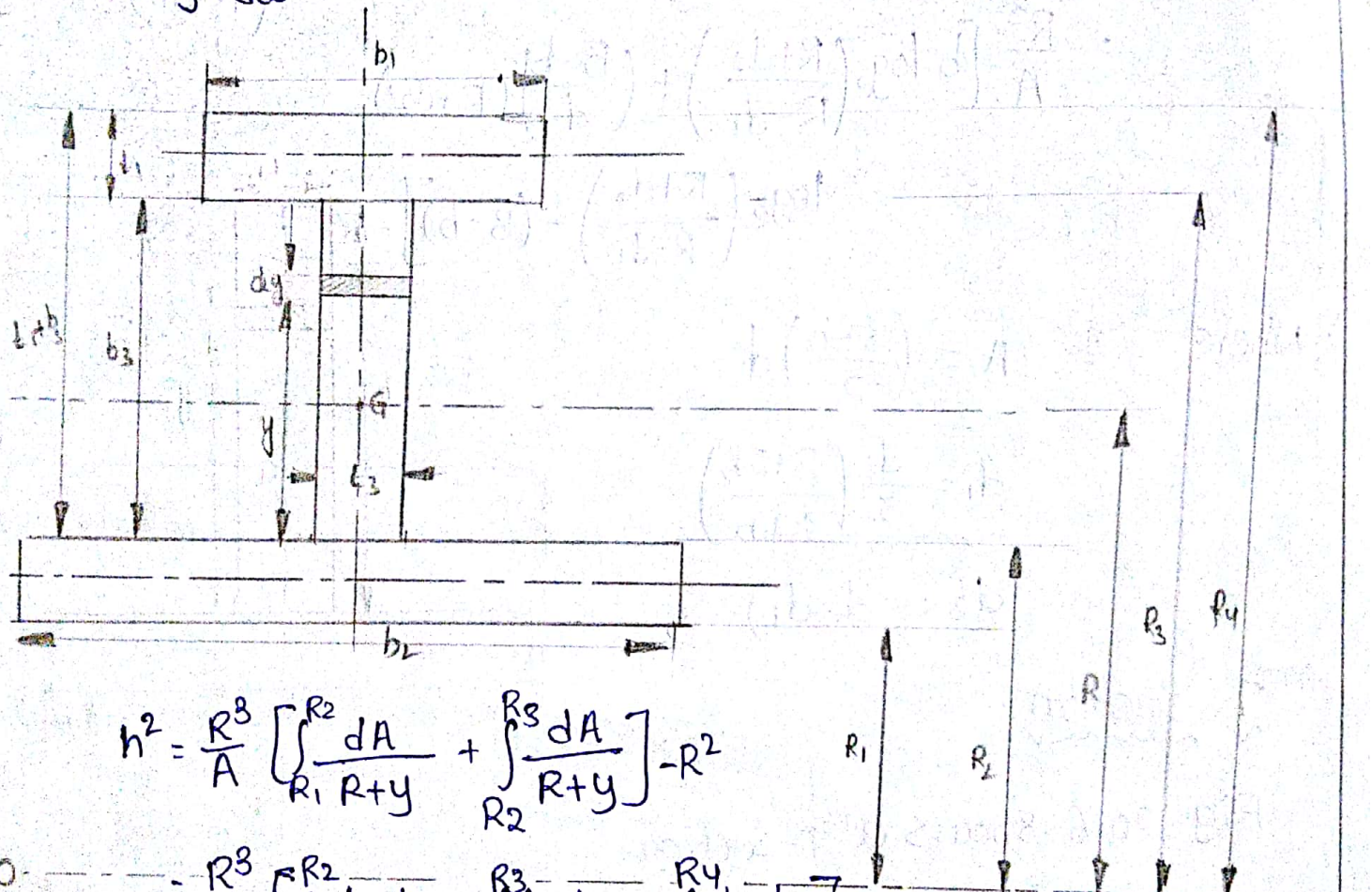
$$\text{Where } A = b_1 t_1 + b_2 t_2$$



I-Section

Consider the I-Section shown in Fig 20.7

Let, $R+y=a$
 $dy=da$



$$h^2 = \frac{R^3}{A} \left[\int_{R_1}^{R_2} \frac{dA}{R+y} + \int_{R_2}^{R_3} \frac{dA}{R+y} \right] - R^2$$

$$= \frac{R^3}{A} \left[\int_{R_1}^{R_2} \frac{b_2 da}{a} + \int_{R_2}^{R_3} \frac{t_3 da}{a} + \int_{R_3}^{R_4} \frac{b_1 da}{a} \right] - R^2$$

$$\text{Or } h^2 = \frac{R^3}{A} \left[b_2 \log_e \frac{R_2}{R_1} + t_3 \log_e \frac{R_3}{R_2} + b_1 \log_e \frac{R_4}{R_3} \right] - R^2$$

Where, $A = b_1 t_1 + b_2 t_2 + b_3 t_3$.