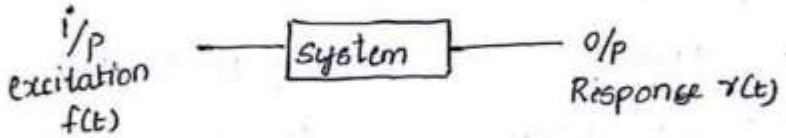
UNIT-4

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

System: A system is defined as set of rules that associates an o/p time function to every i/p time function.

(or)

A system is an interconnection of elements which produces expected o/p for available i/p.



$f(t) \longrightarrow r(t)$

→ System is an mathematical operator which maps i/p into o/p

Classification of system

↓

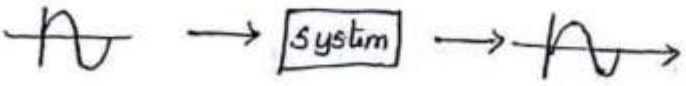
- continuous time system
- Discrete time system

↓

1. Static & Dynamic systems
2. Linear & Non-Linear
3. Time invariant & Time variant
4. Linear TI & LTIV
5. Stable system
6. Casual & non-causal systems.

(i) continuous time systems

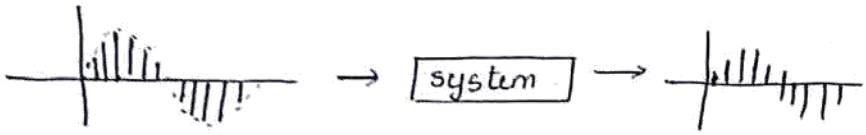
→ A continuous time system operates on a continuous time i/p signal to produce a continuous time o/p signal



(ii) Discrete time systems:

A discrete time system operates on a discrete time i/p signal to produce a discrete time o/p signal.

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Classifications :

(i) Static and Dynamic Systems:

→ A static system or system is said to be static if its o/p at any instant depends only on present values of i/p.

Ex: $y(t) = ax(t)$

at $t=0$ $y(0) = ax(0)$

at $t=1$ $y(1) = ax(1)$

(ii) $y(t) = a^2 x(t)$

at $t=0$ $y(0) = a^2 x(0)$

at $t=1$ $y(1) = a^2 x(1)$

→ A system is said to be dynamic if its o/p depends on present & past values of i/p.

Ex: $y(t) = x(t-1) + x(t-2) + x(t)$

at $t=2$

$$y(2) = x(2-1) + x(2-2) + x(2) = x(1) + x(0) + x(2)$$

$x(1)$: past; $x(2)$: present

(ii) Linear and Non Linear Systems :

→ A system is said to be linear if its satisfies the superposition principle.

→ It states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of o/p's of the system to each of the individual i/p signal.

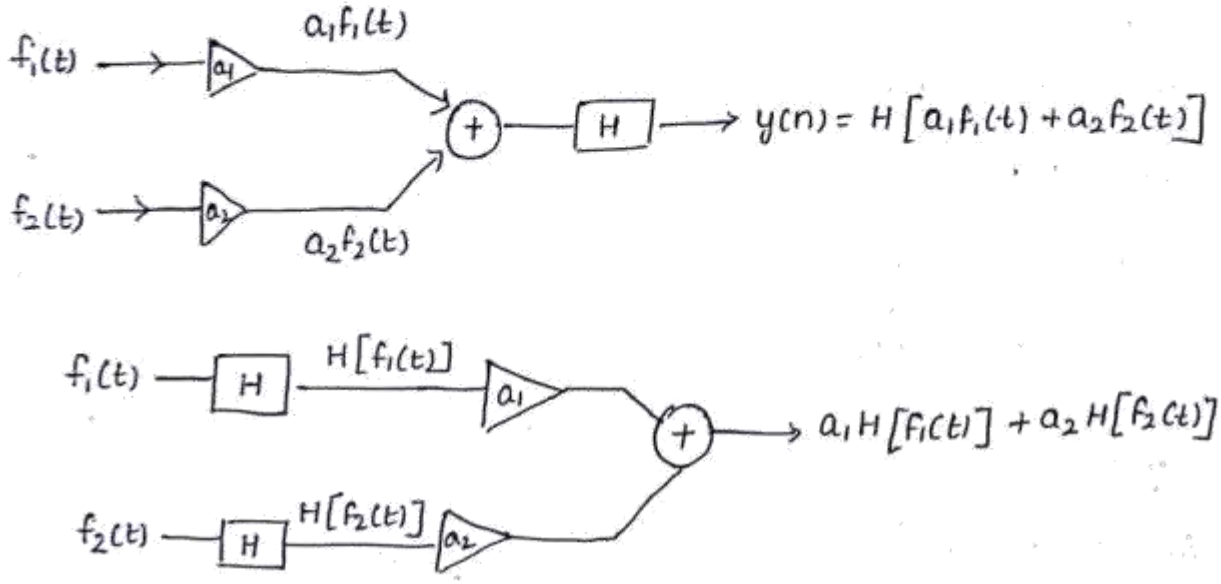
$$H[a_1 f_1(t) + a_2 f_2(t)] = a_1 H[f_1(t)] + a_2 H[f_2(t)]$$

where a_1, a_2 are weighted constants.

$a_1 f_1(t) \xrightarrow{\text{Response}} a_1 H[f_1(t)]$

$a_2 f_2(t) \xrightarrow{\text{Response}} a_2 H[f_2(t)]$

$$H[a_1 f_1(t) + a_2 f_2(t)] \longrightarrow a_1 H[f_1(t)] + a_2 H[f_2(t)]$$



Block diagram.

→ Any system which does not obey the above principle is called as non-linear systems.

Check for Linearity:

Procedure:

1. Apply different i/p's separately and get the o/p.
2. Apply different i/p's simultaneously and get the output.
3. If both outputs are same it is linear otherwise non-linear.

Ex:

(i) $y(t) = 4 \sin t\, x(t)$

Step 1: $y_1(t) = 4 \sin t\, x_1(t)$

$$y_2(t) = 4 \sin t\, x_2(t)$$

$$y_1(t) + y_2(t) = 4 \sin t\, [x_1(t) + x_2(t)]$$

Step 2: $y(t) = 4 \sin t\, [x_1(t) + x_2(t)]$

$$S_1 = S_2$$

(2) $y(t) = ax(t)$

Sol S_1: $y_1(t) = a x_1(t)$

$y_2(t) = a x_2(t)$

$y(t) = a x_1(t) + a x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

S_2: $y(t) = a[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

(4) $y(t) = e^{x(t)}$

Sol S_1: $y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$

$y(t) = e^{x_1(t)} + e^{x_2(t)}$

$y(t) = e^{[x_1(t) + x_2(t)]}$

S_2: $e^{x_1(t)} \cdot e^{x_2(t)}$

$S_1 \neq S_2$ (Non-Linear)

(6) $y(t) = x(t - t_0)$

Sol S_1: $y_1(t) = x_1(t - t_0)$

$y_2(t) = x_2(t - t_0)$

$y(t) = x_1(t - t_0) + x_2(t - t_0)$

S_2: $y(t) = x_1(t - t_0) + x_2(t - t_0)$

$S_1 = S_2$ (Linear)

(8) $y(t) = x(t+1)e^{-t}$

Sol S_1: $y_1(t) = x_1(t+1)e^{-t}$; $y_2(t) = x_2(t+1)e^{-t}$

$y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

S_2: $y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

$S_1 = S_2$ (Linear)

(1) $y(t) = 4x(t) + 2\frac{dx(t)}{dt}$ → Linear.

(3) $y(t) = x^2(t)$

Sol S_1: $y_1(t) = x_1^2(t)$

$y_2(t) = x_2^2(t)$

$y(t) = x_1^2(t) + x_2^2(t)$

S_2: $y(t) = [x_1(t) + x_2(t)]^2$ →

$S_1 \neq S_2$ (Non-Linear)

(5) $y(t) = t\,x(t)$

Sol $y_1(t) = t\,x_1(t)$

S_1: $y_2(t) = t\,x_2(t)$

$y(t) = t[x_1(t) + x_2(t)]$

S_2: $y(t) = t[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

(11) $y(t) = x(t^2)$

Sol S_1: $y_1(t) = x_1(t^2)$

$y_2(t) = x_2(t^2)$

$y(t) = x_1(t^2) + x_2(t^2)$

S_2: $y(t) = x_1(t^2) + x_2(t^2)$

$S_1 = S_2$ (Linear)

(7) $y(t) = 3x(t+3)$

Sol S_1: $y_1(t) = 3x_1(t+3)$

$y_2(t) = 3x_2(t+3)$

$y(t) = 3[x_1(t+3) + x_2(t+3)]$

S_2: $3[x_1(t+3) + x_2(t+3)]$

$S_1 = S_2$ (Linear)

(12) $y(t) = Ax(t) + B$

Sol $y_1(t) = Ax_1(t) + B$

$y_2(t) = Ax_2(t) + B$

$y(t) = A[x_1(t) + x_2(t)] + 2B$

$y(t) = A[x_1(t) + x_2(t)] + B$

$S_1 \neq S_2$ = Non-Linear

(9) $y(t) = \cos[x(t)]$

Sol S_1: $y_1(t) = \cos[x_1(t)]$; $y_2(t) = \cos[x_2(t)]$

$y(t) = \cos[x_1(t)] + \cos[x_2(t)]$

S_2: $\cos[x_1(t) + x_2(t)]$

$S_1 \neq S_2$ (Non-Linear)

(10) $y(t) = k\Delta x(t)$ where $\Delta x(t) = [x(t+1) - x(t)]$

Sol S_1: $y_1(t) = k[x_1(t+1) - x_1(t)]$; $y_2(t) = k\Delta x_2(t)$

$y(t) = [x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)] \cdot k$

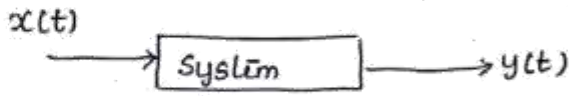
S_2: $y(t) = k[x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)]$

$S_1 = S_2$ (Linear)

Time Variant And Time Invariant Systems

→ A system is said to be time invariant if the system does not depend on time i.e system delay is not function of time.

Ex:



$x(t) \longrightarrow y(t)$

$x(t-t_0) \longrightarrow y(t-t_0)$

→ A time shift to in the input results in the same amount of time shift in the o/p but the waveshape does not change.

i.e the i/p and/ to o/p characteristics does not change with time.

→ Any system which doesnot obey the above principle is called as time varying system.

→ An electrical system is said to be time invariant if its component values (R, L, C) does not change with time.

Check for time Invariant:

1. Shift the i/p only and get the o/p.
2. Shift the entire system and get the o/p.
3. If both steps are identical for o/p then it is time invariant system.

Ex!

(1) $y(t) = 4x(t)$

Sol $S_1: y(t) = 4x(t-1)$

$S_2: y(t-1) = 4x(t-1)$

$\rightarrow S_1 = S_2$ (TIV)

(2) $y(t) = 4t\,x(t)$

Sol $S_1: y(t) = 4t\,x(t-1)$

$S_2: y(t-1) = 4(t-1)(x(t-1))$

$\rightarrow S_1 \neq S_2$ (TV)

(3) $y(t) = ax(t)$

Sol $S_1: y(t) = ax(t-1)$

$S_2\ y(t-1) = ax(t-1)$

$\rightarrow S_1 = S_2$ (TIV)

(4) $y(t) = ax(t) + b$

Sol $S_1: y(t) = ax(t-1) + b$

$S_2: y(t-1) = ax(t-1) + b$

$S_1 = S_2$ (TIV)

(5) $y(t) = 5t\,[x(t)]^2$

Sol $S_1: y(t) = 5t\,[x(t-1)]^2$

$S_2: y(t-1) = 5(t-1)[x(t-1)]^2$

$S_1 \neq S_2$ (TV)

(6) $y(t) = x(t+1)e^{-t}$

Sol $S_1: y(t) = x(t+1-1)\,e^{-t} = x(t)e^{-t}$

$S_2: y(t-1) = x(t+1-1)e^{-(t-1)}$

$= x(t) \cdot e^{-t}e^{1}$ → constant

$S_1 = S_2$ (TIV)

⑦ $y(t) = x(t+3)$

Sol S_1: $y(t) = x(t+3-1)$

S_2: $y(t-1) = x(t+3-1)$

$= x(t+2)$

$\therefore S_1 = S_2$ (TIV)

⑧ $y(t) = x^2(t)$

Sol S_1: $y(t) = x^2(t-1)$

S_2: $y(t-1) = x^2(t-1)$

$\therefore S_1 = S_2$ (TIV)

⑨ $y(t) = e^{x(t)}$

Sol S_1: $y(t) = e^{x(t-1)}$

S_2: $y(t-1) = e^{x(t-1)}$

$S_1 = S_2$ (TIV)

Linear Time Invariant System (LTI):

→ Any system which obeys the linearity and time invariant property is called as LTI system.

Linear Time Variant System (LTV):

→ Any system which obeys the linearity and does not obey time invariant-property is called LTV system.

Ex: $y(t) = a\,x(t)$

Linearity: $y_1(t) = a x_1(t)$; $y_2(t) = a x_2(t)$

$y(t) = a x_1(t) + a x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

S_2: $y(t) = a[x_1(t) + x_2(t)]$.

$\therefore S_1 = S_2$.

T.I: $y(t) = a x(t)$

S_1: $y(t) = a x(t-1)$

S_2: $y(t-1) = a x(t-1)$ → $S_1 = S_2$ (TIV)

$\therefore$ It is a linear time invariant system (LTI)

Q4:

(2) $y(t) = t\,x(t)$ → LTI

(3) $y(t) = a\,x(t) + b$ → NLTI

(4) $y(t) = a x^2(t)$ → NLTI

(5) $y(t) = e^{x(t)}$ → NLTI

(6) $y(t) = x(t - t_0)$ → LTI

Stable System:

→ System is absolutely integrable

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Causal And Non Causal Systems:

→ A system is said to be causal if o/p $y(t_0)$ depends only on the values of i/p $x(t)$ at $t < t_0$ { present, i/p, past i/ps, past o/ps }

{ $x(t) = 0$, for $t < 0$
non causal $t \leq 0$, or $t \leq 0$ and $t > 0$
$x(t) \neq 0$ for $t < 0$

Ex: $y(t) = 4x(t-1)$

$y(2) = 4x(2-1) \Rightarrow 4x(2)$

$y(t) = 4x(t-1) + x(t)$

$y(2) = 4x(1) + x(2)$

→ A system is said to be non-causal if the o/p depends on future values of i/p i.e future i/ps & o/ps.

Ex: $y(t) = 4x(t+1)$

$y(2) = 4x(3)$

Examples whether it is causal & Non causal:

(1) $y(t) = K[x(t+1) - x(t)]$

$y(0) = K[x(1) - x(0)]$ → Non causal

(2) $y(t) = 3x(t+3)$

$y(0) = 3x(3)$ → Non causal

(3) $y(t) = (t+3)x(t-3)$

$y(0) = (0+3)x(0-3)$

$= 3x(-3)$ → causal

(6) $y(t) = x(2t)$ → Non causal

(7) $y(t) = x(t) - x(t-1)$ → causal

(8) $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$

$= x(t) + z(\lambda)]_0^t \Rightarrow$ Causal.

At $t=0, t=1, t=2$

(4) $y(t) = x(t) + 3x(t+4)$

when $t=0$, $y(0) = x(0) + 3x(4)$

when $t=1$, $y(1) = x(1) + 3x(5)$

So here response at $t=0$, $y(0)$ depends on the present i/p & future i/p

Here system is noncausal.

(5) $y(t) = x(t^2)$

$t=-1$, $y(-1) = x(1)$ → future

$t=0$, $y(0) = x(0)$ → present

$t=1$, $y(1) = x(1)$ → present

$t=2$, $y(2) = x(4)$ → future

Non causal.

Except at $t=0$, $t=1$, the response of any value of t depends on future i/p.

Impulse Response:

The response of a system for an impulse i/p is called a impulse response of the system and it is denoted by h(t)

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t) \rightarrow h(t)$$

→ Every system is characterised by its impulse response.

$$\text{i/p } f(t) \longrightarrow \boxed{h(t)} \longrightarrow \text{o/p } r(t)$$

Response of a System for an arbitary i/p:

Response of $\delta(t) \rightarrow h(t)$

$$\delta(t-t_0) \rightarrow h(t-t_0)$$

$$\delta(t) + \delta(t-t_0) = h(t) + h(t-t_0)$$

The response of a system for a given i/p f(t) is determined by using superposition principle.

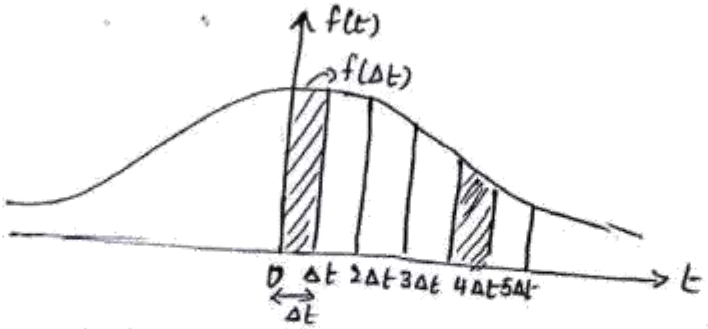
step 1: Resolve the i/p function interms of impulse functions.

step 2: Determine individually the response of LTI system for impulse function.

step 3: Find the sum of individual responses which will become the overall response r(t).

Representation of a function f(t) in terms of an impulse function:

Here the function f(t) is a impulse train function.





→ area $f(5\Delta t) \times \Delta t$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\,\Delta t \cdot \delta(t-n\Delta t)$$

$$f(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta t)\, \Delta t\, \delta(t - n\Delta t)$$

The rectangular of width Δt & height $f(n\Delta t)$ and area under the rectangles is $\Delta t \cdot f(n\Delta t)$ and this n^{th} element approached a delta function of strength $f(n\Delta t)\,\Delta t$ located at $t = n\Delta t$. and this delta function is represented as $f(n\Delta t)\,\Delta t\,\delta(t - n\Delta t)$

$$\boxed{f(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t \cdot \delta(t - \Delta t)}$$

As $\Delta t \to 0$, the n^{th} element may be considered.

2) Determination of $r(t)$ for the input $f(t)$:

Let $h(t)$ be the impulse response of the system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

then $\delta(t) \to h(t)$

$\delta(t - n\Delta t) \longrightarrow h(t - A\Delta t)$

$f(n\Delta t)\,\delta(t - n\Delta t) \longrightarrow f(n\Delta t) \cdot h(t - n\Delta t)$

$f(n\Delta t) \cdot \Delta t\, \delta(t - n\Delta t) \longrightarrow f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$

$$\lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, \delta(t - n\Delta t) \longrightarrow \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$$

$$f(t) \longrightarrow \boxed{\text{system}} \longrightarrow r(t)$$

$$r(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$$

$\Delta t \to 0$ means summation becomes integration.

$$r(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t - \tau)\, d\tau$$

$$r(t) = f(t) \circledast h(t)$$

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow r(t) \to f(t) \circledast h(t).$$

i.e if the response of a system is known for an impulse fn, then response to any other function f(t) can be obtained from the above eqn.

→ An unit impulse function is called as a Test function and it is used to characterise a system.

$$r(t) = f(t) \otimes h(t)$$

In frequency domain $r(t) \xleftrightarrow{FT} R(w)$

$$f(t) \xleftrightarrow{FT} F(w)$$

$$h(t) \xleftrightarrow{FT} H(w)$$

Using convolution property

$$f(t) \otimes h(t) = F(w) \otimes H(w)$$

$$R(w) = F(w) \cdot H(w)$$

$$\boxed{H(w) = \frac{R(w)}{F(w)}}$$

→ When $F(w) = 1$; i.e i/p is unit impulse $H(w) = R(w)$

→ Transfer function $H(w)$ of a system is defined as the transform of the response of a system where the i/p is unit impulse function.

$$H(w) = |H(w)| e^{j\theta(w)}$$

$|H(w)|$ → Amplitude response of the system.

$\theta(w)$ → phase response of the system.

$$\boxed{\begin{aligned} \ln[H(w)] &= \ln[|H(w)|] + j\theta(w) \\ &= L(w) + j\theta(w) \end{aligned}}$$

$L(w)$ → Gain of the system

$\theta(w)$ → phase shift introduced by system.

Note: An impulse function contains all frequencies in equal amount so we can use it as a test function.

$$H(w) = \frac{R(w)}{F(w)} \rightarrow \text{Transfer fn of LTI system.}$$

$$FIT[h(t)] \quad h(t) = I \cdot FIT[H(w)]$$

FILTER CHARACTERISTICS OF LINEAR SYSTEMS :

IDEAL LOW PASS FILTERS :

→ It transmits all the signals below certain frequency 'B' Hz without any distortion.

→ The range of frequencies from 0Hz to 'B' Hz is called passband of lowpass filter.

→ The frequency 'B' Hz is called cut-off frequency of the ideal lowpass filter.

→ The transfer function of ideal low pass filter can be written as

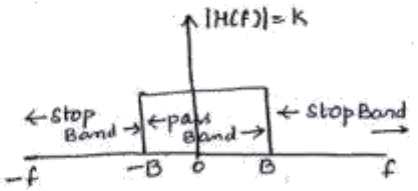
$$H(f) = K e^{-j2\pi f t_0} \quad ; -B \le f \le B$$
$$= 0 \quad ; |f| > B.$$

K = amplitude is assumed to be unity.

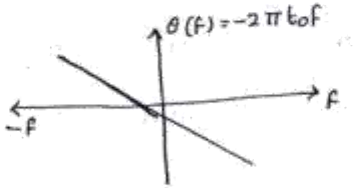
→ By K=1 in above eqn

$$H(f) = e^{-j2\pi f t_0} \quad ; -B \le f < B$$
$$= 0 \quad ; |f| > B$$

→ By inverse fourier transform, h(t) can be obtained for ideal LPF



a) Magnitude response



(b) phase response

$$h(t) = \int_{-B}^{B} e^{-j2\pi f t_0} \cdot e^{j2\pi f t} \, df$$

$$= \int_{-B}^{B} \left[e^{j2\pi f (t-t_0)} \right] df = \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi f(t-t_0)} \right]_{-B}^{B}$$

$$= \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)} \right]$$

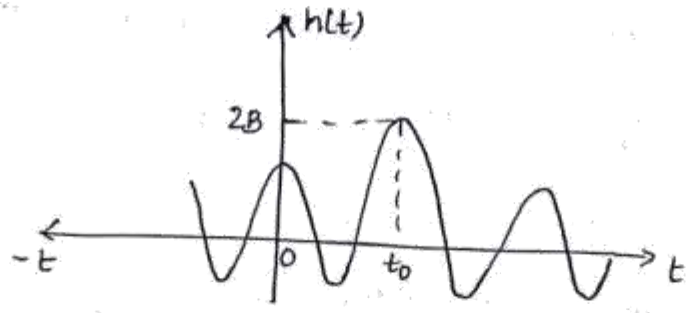
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$$= \frac{1}{\pi(t-t_0)} \left[\frac{e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}}{2j} \right]$$

$$= \frac{1}{\pi(t-t_0)} \sin\left[2\pi B(t-t_0)\right]$$

$$h(t) = 2B \left(\frac{\sin[2\pi B(t-t_0)]}{2\pi B(t-t_0)} \right) = 2B \operatorname{sinc}[2B(t-t_0)]$$

Response



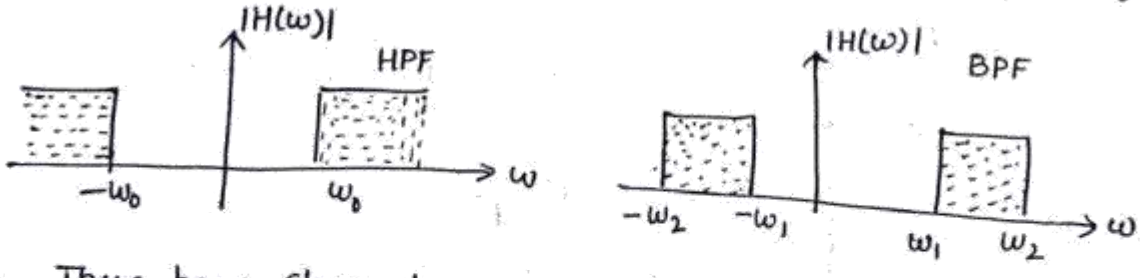
→ Figure shows that impulse response exists for negative values of 't'. But actually unit impulse is applied at t=0 always.

→ Practically it is impossible to implement such a system.

OTHER IDEAL FILTERS SUCH AS HPF, BPF etc.,

→ In realizability of ideal LPF its response begins before input is applied and hence it is not physically realizable.

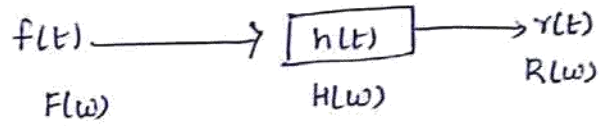
→ 11^{th} HPF, BPF ideal have frequency response as shown in figure



→ These have sharp transition in frequency response.

→ All ideal filters are physically not realizable since their impulse response is non-causal.

INTRODUCTION FOR FILTER CHARACTERISTICS:



$$R(w) = F(w).H(w)$$

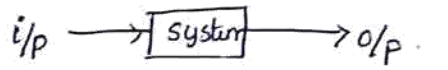
→ The spectrum of o/p is F(w).H(w) i.e the system acts as a kind of filter to various frequency components.

→ Some frequency components are boosted in strength and some are attenuated and some remain unaffected.

→ i.e each freq component undergoes a different amount of phase shift i.e the modification is carried out according to H(w).
↳ acts as waiting fn for two different frequencies.

DISTORTIONLESS TRANSMISSION THROUGH SYSTEM:

→ It means output signal is an exact replica of the i/p signal.



→ The difference between i/p and o/p of such system is that

1. Amplitude of the o/p signal may increase or decrease by some factor w.r. to i/p.
2. The o/p sgl may be delayed in time w.r. to i/p sgl because of system delay.

→ o/p sgl y(t) can be written in terms of i/p x(t) as

$$y(t) = k\,x(t-t_0)$$

k: constant, Represents change in amplitude

t_0: time delay in transmission of signal through a system.

By taking fourier transform

$$Y(f) = F[y(t)] = F\{k\,x(t-t_0)\}$$

From time shifting property of FT

$$Y(f) = k\,X(f)\,e^{-j2\pi f t_0}$$

Transfer fn $H(f) = \frac{Y(f)}{X(f)}$

$$\therefore H(f) = \frac{Y(f)}{X(f)} = k \cdot e^{-j2\pi f t_0}$$

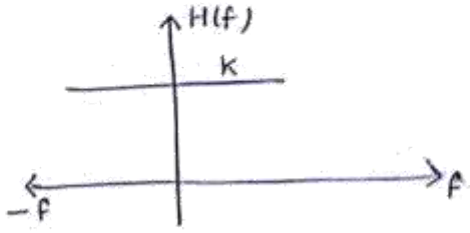
Magnitude of transfer fn independent of frequency.

→ Transfer function has constant amplitude at all frequencies. The phase shift is

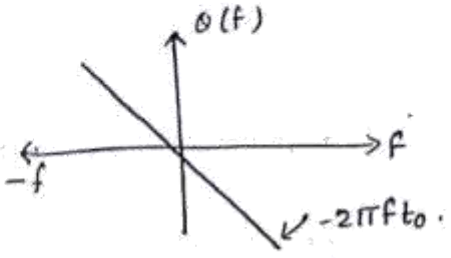
$$\theta(f) = -2\pi f t_0$$

$$= (-2\pi t_0) f$$

→ phase shift is linearly proportional to frequency.



(a) Amplitude Spectrum



(b) phase spectrum passing through origin

→ By considering simple example

Let there be signal in time domain as

$$x(t) = \cos(2\pi f t)$$

Let the o/p sgl be same in amplitude but shifted in time by t_0 sec

$$y(t) = \cos[2\pi f(t - t_0)]$$

$$\therefore y(t) = \cos(2\pi f t - 2\pi f t_0) = \cos(2\pi f t - \theta(f))$$

$\therefore$ phase shift of $y(t)$ is

$$\theta(f) = -2\pi f t_0.$$

which is proportional to frequency 'f'.

Two types:

(i) Amplitude distortion

(ii) phase distortion.

AMPLITUDE DISTORTION

→ This distortion occurs when $|H(\omega)|$ is not constant over frequency band of interest and the frequency components present in I/P sgl are transmitted with different gain and attenuation.

PHASE DISTORTION:

→ This distortion occurs when phase of $H(\omega)$ is not linearly changing with time and different frequency components in I/P are subjected to different time delays during transmission.

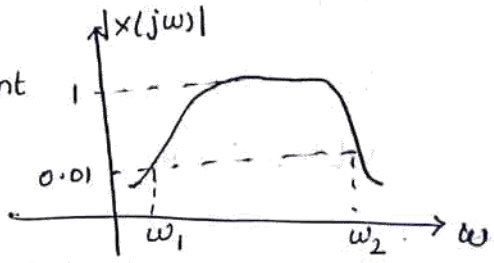
SIGNAL BANDWIDTH:

The band of frequencies that contains most of signal energy is called B.W of signal denoted by fm.

→ It is the range of significant signal frequencies which are present in the signal.

→ observe in the waveform $x(t)$ has significant frequencies from ω_1 to ω_2.

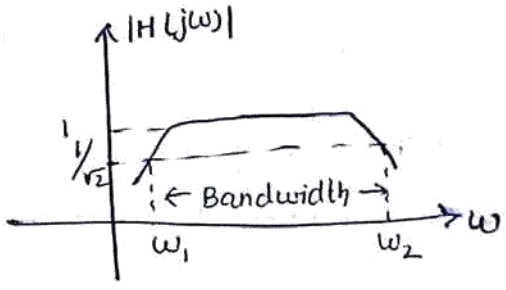
→ The B.W of this signal is $\omega_2 - \omega_1$



→ All the physically obtained signals have limited bandwidth.

SYSTEM BANDWIDTH

→ The B.W of a system is defined as range of frequencies over which $|H(\omega)|$ remains with in $1/\sqrt{2}$ times of its mid-band value. for distortionless transmission the system must have infinite B.W but physical system are limited to finite BW.



→ So a system with finite BW can provide distortionless transmission for a band limited signal if $|H(\omega)|$ remains constant over BW of the signal.

→ The range of frequencies for which magnitude $|H(j\omega)|$ of the systems remains within $\frac{1}{\sqrt{2}}$ of its maximum value

CASUAL SYSTEMS:

→ A system is said to be causal if $h(t) = 0$ if $t < 0$

$h(t-t_0) = 0$; $t < t_0$

i.e if i/p is zero for $t < t_0$, then o/p is also zero for $t < t_0$.

→ Any system which does not obey the above rule is non-causal system.

→ If two i/p to a causal system are equal upto some time 't0' then corresponding o/p must be equal upto that time instant.

POLY-WIENER CRITERION

→ This gives the condition for causality in frequency domain or in other words the frequency domain equivalent of causal system i.e $H(\omega)$.

→ Consider a system with transfer function $H(\omega)$, the necessary and sufficient condition for $H(\omega)$ to be transfer function of causal fn is

$$\int_{-\infty}^{\infty} \frac{|\ln(H(j\omega))|}{1+\omega^2} d\omega < \infty \rightarrow (1)$$

provided $|H(j\omega)$ is square integral.

$$\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega < \infty \rightarrow (2)$$

→ This is poly-wiener criteria. If condition (2) is not satisfied then the condition (1) is neither necessary nor sufficient.

PHYSICAL REALIZABILITY:

→ A system is said to be physically realizable if it obeys the causal condition

i.e $h(t) = 0$ for $t < 0$.

Ex: $H(\omega) = \frac{1}{1+j\omega}$

$h(t) = e^{-t} u(t)$

$= 0$ for $t < 0$

So the above system for transfer fn is realizable in freq. domain.

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{H(\omega)^2} d\omega < \infty$$

→ The frequency domain statements can be interpreted as $|H(\omega)|$ of a physically realizable system may be zero for some discrete frequency but it can never be zero for a finite band of frequencies.

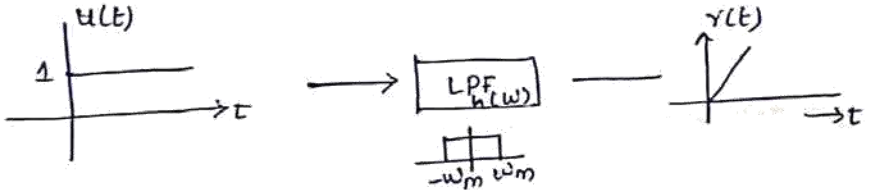
→ $H(\omega)$ for a realisable system cannot decay faster than a function of exponential order.

Ex: A system with T.F $e^{-\omega}$ is realisable. whereas $e^{-\omega^2}$ is not as it decay faster

RELATIONSHIP BETWEEN RISE TIME AND BANDWIDTH:

→ If a unit step fn u(t) is applied to an ideal LPF, the o/p will show a gradual rise instead of a sharp rise in the i/p.

→ The rise time (t_r) is the time required by the response to reach its final value from initial value.



Transfer function of ideal low pass filter is

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

$$= G_1(\omega)\, e^{-j\omega t_0}$$

↓

Rectangular pulse with magnitude K.

for $-B < f \leq B$ i.e $-\omega_m \leq \omega \leq \omega_m$ where $\omega_m = 2\pi B$.

and $\theta(\omega) = -2\pi f t_0 = -\omega t_0$.

→ Fourier transform of unit step fn u(t)

$$FT\{u(t)\} \Rightarrow U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

→ Fourier transform of response $R(\omega)$, input and $H(\omega)$ related as

$$R(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] H(\omega) = \pi\delta(\omega).H(\omega) + \frac{1}{j\omega} H(\omega).$$

$\delta(\omega)$ exists only for $\omega=0$ and $H(\omega)|_{\omega=0}=1$

$$R(\omega)=\pi\delta(\omega)+\frac{1}{j\omega}H(\omega)$$

By taking IFT for above eqn

$$r(t)=IFT[R(\omega)]=IFT\left\{\pi\delta(\omega)+\frac{1}{j\omega}H(\omega)\right\}$$

$$=IFT\left\{\pi\delta(\omega)+\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\right\}\quad(\because H(\omega)=G(\omega)e^{-j\omega t_0})$$

Inverse fourier transform of $\pi\delta(\omega)$ is $\frac{1}{2}$.

$\left\{\because 1\longrightarrow 2\pi\delta(\omega),\ \frac{1}{2}\longleftarrow\pi\delta(\omega)\right\}$

$$r(t)=\frac{1}{2}+IFT\left\{\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\right\}$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\cdot e^{j\omega t}d\omega$$

we know $G(\omega)=1$ for $-\omega_m\le\omega\le\omega_m$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{e^{j\omega(t-t_0)}}{j\omega}d\omega$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\cos\omega(t-t_0)+j\sin\omega(t-t_0)}{j\omega}d\omega$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\cos\omega(t-t_0)}{j\omega}d\omega+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}d\omega.$$

$\frac{\cos\omega(t-t_0)}{j\omega}$ = zero for its odd term

$\frac{\sin\omega(t-t_0)}{\omega}$ → even

Si(x) is an odd fn

Si(-x) = -Si(x)

$$r(t)=\frac{1}{2}+\frac{1}{2\pi}\times2\int_0^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}d\omega=\frac{1}{2}+\frac{1}{\pi}\int_0^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}d\omega$$

$$=\frac{1}{2}+\frac{1}{\pi}\left[Si\,\omega(t-t_0)\right]_0^{\omega_m}$$

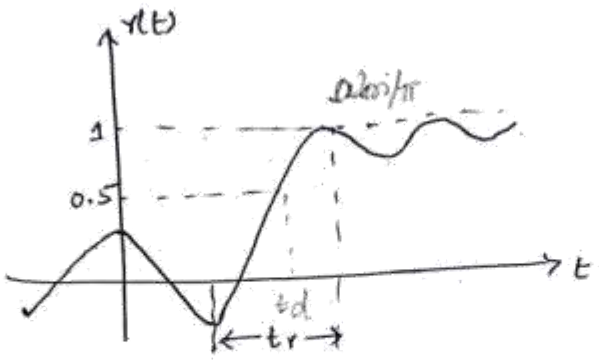
$$=\frac{1}{2}+\frac{1}{\pi}Si\,\omega_m(t-t_0)\longrightarrow\text{sine integral}$$

$$\rightarrow\left.\frac{dy(t)}{d}\right|_{t=t_0}=\frac{1}{\pi}\cos[\omega_m(t-t_0)].\ \omega_m$$

The rise time is given as $t_r=\frac{2\pi}{\omega_m}=\frac{1}{B}$

(ω_m: cut off frequency of LPF)

$$\frac{1}{t_r}=\frac{\omega_m}{\pi}\Rightarrow t_r=\frac{\pi}{\omega_m}$$



$\omega_c \to \infty$

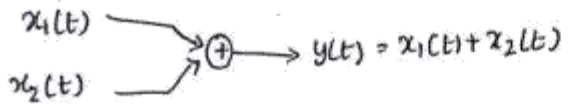
$y(t) = 1$

$\omega_c \to -\infty$

$y(t) = 0$

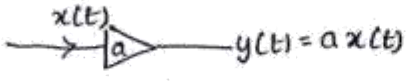
Note: { Elements of block diagram }

① Adder:



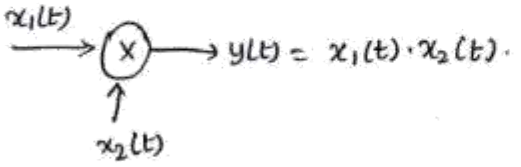
which performs the addition of two signal sequences to form sum

② constant multiplier:



It represents applying a scale factor on i/p x(t).

③ Signal multiplier:



The multiplication of two signal to form product sequence.

PROBLEMS:

① The impulse response of continuous time system is given as

$$h(t) = \frac{1}{RC} e^{-t/RC} \cdot u(t)$$

Determine the frequency response & plot the magnitude phase plots.

Sol Take FT

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} \cdot e^{-t/RC} \cdot u(t)\, e^{-j\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC} \cdot e^{-j\omega t} dt \quad \left(\because u(t) = \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ otherwise} \end{cases} \text{ step.} \right.$$

$$= \frac{1}{RC} \int_0^\infty e^{-t\left(j\omega + \frac{1}{RC}\right)} dt$$

$$= \frac{1}{RC} \left(- \frac{1}{j\omega + \frac{1}{RC}} \right) \left[e^{-t(j\omega + 1/RC)} \right]_0^\infty$$

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}.$$

Magnitude & phase

$$H(\omega) = \frac{1}{1+j\omega RC} \times \frac{1-j\omega RC}{1-j\omega RC} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + j \frac{-\omega RC}{1+(\omega RC)^2}$$

$$|H(\omega)| = \left\{ \frac{1}{[1+(\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1+(\omega RC)^2]^2} \right\}^{1/2}$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\angle H(\omega) = \tan^{-1} \left\{ \frac{(-\omega RC)/1+(\omega RC)^2}{1+[1+(\omega RC)^2]} \right\} = -\tan^{-1}(\omega RC)$$

If $RC = 1$, $|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$; $\angle H(\omega) = -\tan^{-1}(\omega)$.



(2) For the system shown find the T.T & impulse response of the system.

$f(t) = e^{-at}$ $t>0$

0 elsewhere ; $Y(\omega) = \frac{1}{\alpha+j\omega}$

Sol

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$R(\omega) = Y(\omega)$

$F(\omega)$ —[ ]— $Y(\omega)$

$$F(t) = e^{-at}$$

$$F(\omega) = \frac{1}{a+j\omega} \quad ; \quad Y(\omega) = \frac{1}{\alpha+j\omega}$$

$$H(\omega) = \frac{1/\alpha+j\omega}{1/a+j\omega} = \frac{a+j\omega}{\alpha+j\omega}$$

$$F^{-1}\left[\frac{a+j\omega}{\alpha+j\omega}\right] \Rightarrow \frac{a+\alpha-\alpha+j\omega}{\alpha+j\omega} = \frac{a-\alpha}{\alpha+j\omega} + \frac{\alpha+j\omega}{\alpha+j\omega}$$

$$= \frac{a-\alpha}{\alpha+j\omega} + 1$$

$$\boxed{h(t) = (a-\alpha)\, e^{-\alpha t} u(t) + \delta(t)}$$

(3) The linear system impulse response is $[e^{-2t} + e^{-3t}]\,u(t)$ find the excitation to produce an o/p of $t \cdot e^{-2t} u(t)$?

Sol

$$h(t) = [e^{-2t} + e^{-3t}]\,u(t)$$

$F(\omega)$ —[$H(\omega)$]— $R(\omega)$

$$r(t) = t \cdot e^{-2t} u(t)$$

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$$F(\omega) = \frac{R(\omega)}{H(\omega)}$$

$$r(t) = t \cdot e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{(2+j\omega)^2} \qquad \left(\because t \cdot e^{-at} u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}\right)$$

$$R(\omega) = \frac{1}{(2+j\omega)^2}$$

$$h(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

$$H(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}$$

$$R(\omega) = \frac{1/(2+j\omega)^2}{\frac{3+j\omega+2+j\omega}{(2+j\omega)(3+j\omega)}} = \frac{1}{2+j\omega} \times \frac{3+j\omega}{5+2j\omega}$$

$$\frac{3+j\omega}{(2+j\omega)(5+2j\omega)} = \frac{A}{2+j\omega} + \frac{B}{5+2j\omega}$$

$$3+j\omega = A(5+2j\omega) + B(2+j\omega)$$

put $3+j\omega = 5A+2B+j\omega(2A+B)$

put $j\omega = 0$; put $j\omega(-2)$

$(3 = 5A+2B) \times 1$

$(1 = 2A+B) \times 2$

$A = 1, \quad B = -1$

$$R(\omega) = \frac{1}{2+j\omega} - \frac{1}{5+2j\omega} = \frac{1}{2+j\omega} - \frac{1}{2[5/2+j\omega]}$$

$$\boxed{r(t) = e^{-2t}u(t) - \frac{1}{2}e^{-5/2t}u(t)}$$

DIFFERENTIAL EQUATION:

→ To obtain frequency response & impulse response.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).$$

differentiation property of FT is

$$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega).$$

$$\sum_{K=0}^{N} a_K (j\omega)^K Y(\omega) = \sum_{K=0}^{M} b_K (j\omega)^K X(j\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{K=0}^{M} b_K (j\omega)^K}{\sum_{K=0}^{N} a_K (j\omega)^K}$$

$H(\omega)$ ↙ system transfer fn.

PROBLEMS:

① The differential equation of system is given as $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$

Determine the frequency response & impulse response.

Sol

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$$

Taking F.T

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = -j\omega X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}$$

$$H(\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+3}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3}$$

$$h(t) = \left[2 \cdot e^{-2t} - 3e^{-3t}\right] u(t)$$

$h(t)$ ↙ impulse response of the system.

$\left\{ \because e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \right\}$

(2) The input voltage to the RC circuit is given by $x(t) = te^{-t/RC}u(t)$ and impulse response of this circuit is given by $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$. Find output $y(t)$

Sol

output $y(t) = x(t) * h(t)$

In frequency domain

$$Y(\omega) = X(\omega)H(\omega)$$

and $H(\omega) = F[h(t)]$

$$H(\omega) = F\left\{\frac{1}{RC}e^{-t/RC}.u(t)\right\}$$

$$= \frac{1}{RC}\cdot\frac{1}{\frac{1}{RC}+j\omega} = \frac{1}{1+j\omega RC}$$

$$X(\omega) = F\left[t.e^{-t/RC}u(t)\right]$$

$$= \int_{-\infty}^{\infty} t.e^{-t/RC}e^{-j\omega t}dt = \frac{1}{\left(\frac{1}{RC}+j\omega\right)^2} = \frac{(RC)^2}{(1+j\omega RC)^2}$$

$$\left[\because te^{-at}u(t) \xleftrightarrow{FT} \left(\frac{1}{a+j\omega}\right)^2\right]$$

$$Y(\omega) = X(\omega).H(\omega)$$

$$= \frac{(RC)^2}{(1+j\omega RC)^2}\cdot\frac{1}{(1+j\omega RC)} = \frac{(RC)^2}{(1+j\omega RC)^3}$$

$$y(t) = F^{-1}\{Y(\omega)\} = F^{-1}\left\{\frac{(RC)^2}{(1+j\omega RC)^3}\right\} = F^{-1}\left\{\frac{(RC)^2}{(RC)^3\left(\frac{1}{RC}+\right.\right.}$$

$$\boxed{y(t) = \frac{1}{RC}\cdot\frac{t^2.e^{-t/RC}}{2}u(t)}$$

1) $h(t) = e^{-5|t|}$

For stability $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} |e^{-5|t|}|\,dt = \int_{-\infty}^{\infty} e^{-5|t|}\,dt$$

$$= \int_{-\infty}^{0} e^{5t}\,dt + \int_{0}^{\infty} e^{-5t}\,dt = \left[\frac{e^{5t}}{5}\right]_{-\infty}^{0} + \left[\frac{e^{-5t}}{-5}\right]_{0}^{\infty}$$

$$= \frac{2}{5} = \text{constant} \;/\; \text{so system is stable.}$$

2) $h(t) = e^{4t}u(t)$

$$= \int_{-\infty}^{\infty} |e^{4t}u(t)|\,dt = \int_{-\infty}^{\infty} e^{4t}u(t)\,dt$$

$$= \int_{0}^{\infty} e^{4t}dt = \frac{e^{\infty}}{4} - \frac{e^{0}}{4} = \infty - \frac{1}{4} = \infty \quad (\text{Unstant}.$$

3) $h(t) = e^{-4t}u(t)$ (stable)

4) $h(t) = t\cos t\,u(t)$ (unstable.

$$\int_{0}^{\infty} t\cos t\,dt$$

5) $h(t) = e^{-t}\sin t\,u(t)$ (stable.

$$= \int_{0}^{\infty} e^{-t}\sin t\,dt$$

→ The system produces the o/p of $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the system.

Sol

$$y(t) = e^{-t}u(t)$$
$$x(t) = e^{-2t}u(t)$$

Consdr standard fourier transform pair $e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$

$$Y(\omega) = \frac{1}{1+j\omega}; \quad X(\omega) = \frac{1}{2+j\omega}.$$

From equation

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/1+j\omega}{1/2+j\omega} = \frac{2+j\omega}{1+j\omega}.$$

Multiply ~~and divide~~ the numerator & denominator by $1-j\omega$

$$H(\omega) = \frac{2+j\omega}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{(2+j\omega)(1-j\omega)}{(1)^2-(j\omega)^2} = \frac{2-2j\omega+j\omega+\omega^2}{1+\omega^2}$$

$$= \frac{2+\omega^2-j\omega}{1+\omega^2} \Rightarrow \frac{2+\omega^2}{1+\omega^2} + j\frac{-\omega}{1+\omega^2}.$$

Magnitude $|H(\omega)| = \left\{\sqrt{\left[\frac{2+\omega^2}{1+\omega^2}\right]^2 + \left[\frac{\omega^2}{1+\omega^2}\right]^2}\right]$

$$|H(\omega)| = \sqrt{\frac{4+\omega^2}{1+\omega^2}} \quad ; \quad \angle H(\omega) = \tan^{-1}\left(\frac{\frac{-\omega}{1+\omega^2}}{\frac{2+\omega^2}{1+\omega^2}}\right)$$
$$= -\tan^{-1}\left(\frac{\omega}{2+\omega^2}\right\}$$

$$\therefore H(\omega) = \frac{2+j\omega}{1+j\omega} \Rightarrow \frac{1+j\omega+1}{1+j\omega} \Rightarrow 1+\frac{1}{1+j\omega}.$$

Inverse fourier transform

$$\boxed{h(t) = IFT\{H(\omega)\} = \delta(t) + e^{-t}u(t)}$$

↓ impulse response.

(1) The transfer function of LPF is given by

$$H(\omega) = \begin{cases} (1+k\cos\omega T)e^{-j\omega T} & ; |\omega| < 2\pi B \\ 0 & ; |\omega| > 2\pi B \end{cases}$$

Determine the output $y(t)$ when a pulse $x(t)$ bandlimited in B is applied at the input.

Sol)

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$= X(\omega)\left[1 + K\cos\omega T\right]e^{-j\omega T}$$

$$= X(\omega)\,e^{-j\omega T} + K\,X(\omega)\cos\omega T\,e^{-j\omega T}$$

we know

$$x(t-T) + x(t+T) \longleftrightarrow 2X(\omega)\cos\omega T$$

$$x(t-T) \longleftrightarrow X(\omega)e^{-j\omega T}$$

$$y(t) = F^{-1}\left[Y(\omega)\right]$$

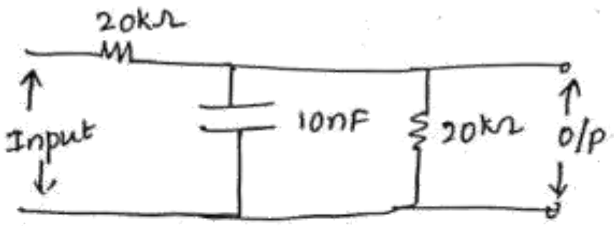
$$= F^{-1}\left[X(\omega)\,e^{-j\omega T} + K\,X(\omega)\,e^{-j\omega T}\cos\omega T\right]$$

$$= x(t-T) + \frac{K}{2}\left[x(t-T-T) + x(t-T+T)\right]$$

$$y(t) = x(t) + \frac{K}{2}\left[x(t-T) + x(t+T)\right]$$

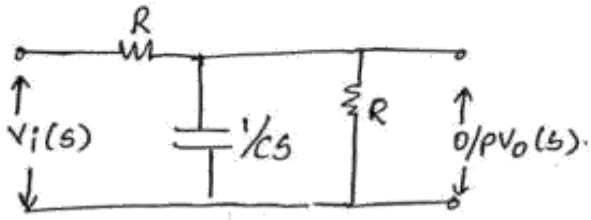
delayed by T.

2) Determine the maximum bandwidth of signals that can be transmitted through low pass RC filter as shown in figure , if over this bandwidth, the gain variation is to be 10% and the phase variation is to be within 7% of ideal characteristics.



Sol

RC network transformed into s-domain representation.



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$= \frac{[R \,||\, (1/Cs)]}{([R \,||\, (1/Cs)] + R]}$$

$$= \frac{(R/Cs) \,/\, [R + 1/Cs]}{[R/Cs] \,|\, [[R + 1/Cs)] + R}$$

$$= \frac{R/(1+SCR)}{[R/(1+SCR)] + R} = \frac{R}{R + R(1+SCR)} = \frac{R}{R(1+1+SCR)}$$

$$H(s) = \frac{1}{2+SCR}.$$

But given $R = 20k\Omega$

$C = 10nF$.

$$= \frac{1}{2+(25/10^4)}$$

$$H(s) = \frac{1}{2+S(10\times10^{-9}\times20\times10^{3})} = \frac{10^4}{2\times10^4+2S} = \frac{1}{2+S(2\times10^{-4})}$$

$$\therefore H(s) = \frac{5000}{S+10000}$$

put $S = j\omega$

$$H(\omega) = \frac{5000}{j\omega + 10000}$$

$$|H(\omega)| = \frac{5000}{\sqrt{\omega^2 + 10000}}$$

$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{10000}\right).$$

At $\omega = 0$, $|H(\omega)|_{\omega=0} = \frac{5000}{10000} = 0.5$.

But there is 10% variation in gain over bandwidth B.

$$H(\omega) = 0.5 - 0.5 \times 10\% = 0.45$$

$$|H(\omega)| = \frac{5000}{\sqrt{B^2 + 10^8}}$$

$$B^2 + 10^8 = \left(\frac{5000}{0.45}\right)^2 \Rightarrow B^2 = 23.46 \times 10^6$$

$$B = 4.84 \text{ KHz}$$

But $B = 2\pi f$

$$f = \frac{B}{2\pi} = \frac{4.84 \times 10^3}{2\pi} = 770.8 \text{ Hz}$$

phase at frequency, $f = 770.8$ Hz

$$\phi(\omega) = -\tan^{-1}\left(\frac{4.84}{10}\right) = -25.83°$$

(2) There are several possible ways of estimating an essential bandwidth of non-band limited signal. For a low pass signal, for example, the essential b.w may

UNIT-4

CONVOLUTION AND CORRELATION OF SIGNALS:

→ Convolution is used to find common area between two signals or two fns.

The convolution f(t) of two time functions $f_1(t)$ and $f_2(t)$ is designed or defined as

$$f(t) = f_1(t) \circledast f_2(t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

(or)

$$= \int_{-\infty}^{\infty} f_2(\tau)\, f_1(t-\tau)\, d\tau$$

→ Convolution is a mathematical operation and multiplication is one of the form of convolution.

Ex: 25
× 25
625

In convolution method

(i) $f_1(\tau)$ & $f_2(\tau)$
↓ 25 ↓ 25

(ii) $f_2(-\tau)$
⇒ 52

(iii) $f_2(t-\tau)$
$f_2(-\tau)$ shifted to right side by t seconds.

→ Here 't' is varied from $-\infty$ to ∞

Convolution Procedure:

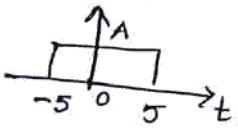
Step 1: $f_1(t) \circledast f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$

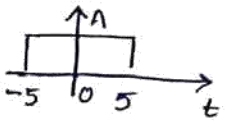
Here the independent variable convolution integral is τ, so replace t by τ to get $f_1(\tau)$ & $f_2(\tau)$

Step 2: $f_1(\tau)$ is first function and $f_2(\tau)$ is the 2nd function. $f_2(-\tau)$ is the mirror image of the $f_2(\tau)$

Step 3: $f_2(t-\tau)$ represents the function $f_2(-\tau)$ shifted to right side by t sec. 't' is varied from $-\infty$ to ∞ and find common area between two functions.

→ The value of convolution obtained at different values of 't' and may be plotted on a graph.

① Find the F.T of 

⊛ 

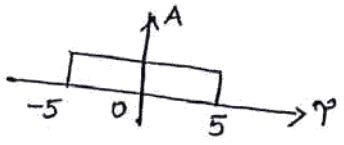
Sol Step 1:

$f_1(t) = A\,G_{10}(t) \longrightarrow f_1(\tau) = A\,G_{10}(\tau)$

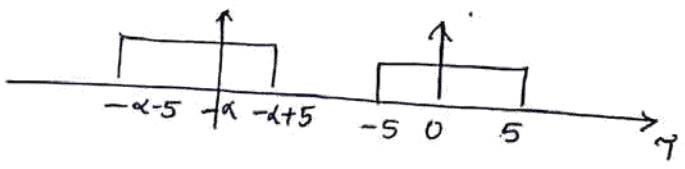
$f_2(t) = A\,G_{10}(t) \longrightarrow f_2(\tau) = A\,G_{10}(\tau)$

Step 2:

$f_2(-\tau) = A\,G_{10}(-\tau) \longrightarrow$ even function

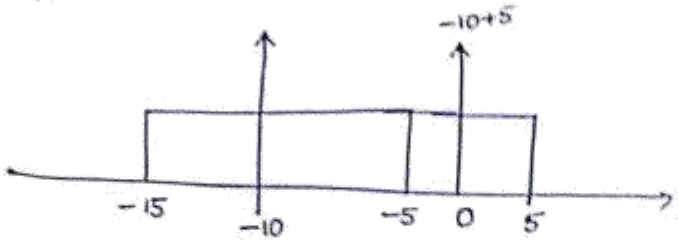


① $f_1(\tau)$ at $t = -\alpha$



$f_1(t) \circledast f_2(t) = 0$

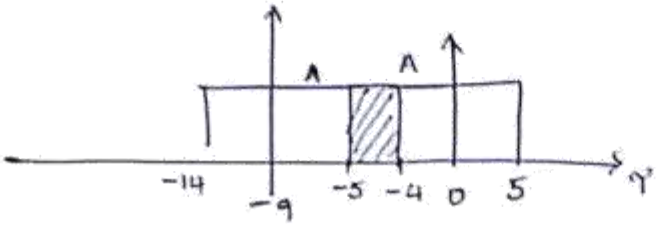
(2) at $t=-10$



$-10+5 = -5$

$-10-5 = -15$

(3) at $t=-9$



$-9-5 = -14$

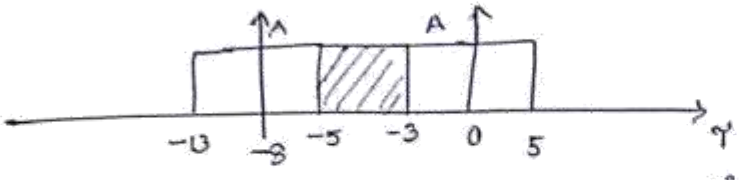
$-9+5 = -4$

$$\Rightarrow \int_{-5}^{-4} A \cdot A \, d\tau = A^2 [\tau]_{-5}^{-4}$$

$$= A^2[-4+5]$$

$$= A^2$$

(4) at $t=-8$



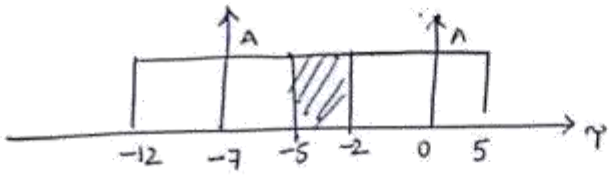
$-8+5 = -3$

$-8-5 = -13$

$$\Rightarrow \int_{-5}^{-3} A \cdot A \, d\tau = A^2 [\tau]_{-5}^{-3}$$

$$= A^2[-3+5] \Rightarrow 2A^2.$$

(5) at $t=-7$



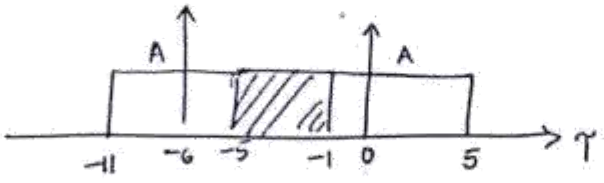
$-7+5 = -2$

$-7-5 = -12$

$$\Rightarrow \int_{-5}^{-2} A \cdot A \, d\tau = A^2 [\tau]_{-5}^{-2}$$

$$= A^2[-2+5] = 3A^2.$$

(6) at $t=-6$



$-6+5 = -1$

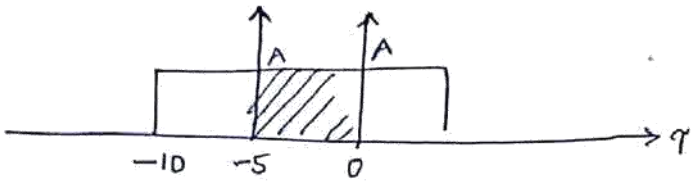
$-6-5 = -11$

$$\Rightarrow \int_{-5}^{-1} A \cdot A \, d\tau \Rightarrow A^2[-1+5] = 4A^2.$$

⑦ at $t=-5$

$-5-5=-10$

$-5+5=0$

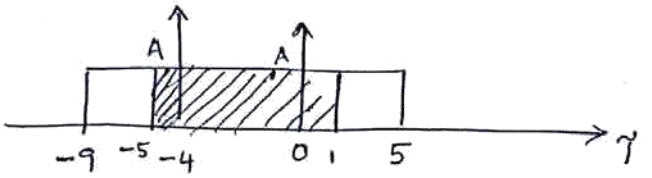


$$\int_{-5}^{0} A\cdot A\, d\gamma \Rightarrow A^2\cdot[\gamma]_{-5}^{0} = A^2\cdot 5$$

⑧ at $t=-4$

$-4+5=1$

$-4-5=-9$

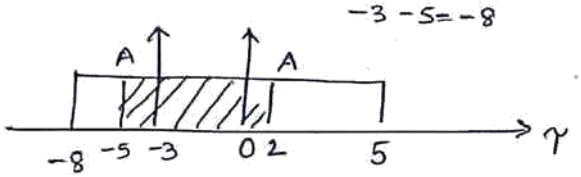


$$\int_{-5}^{1} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{1} = 6A^2.$$

⑨ at $t=-3$

$-3+5=+2$

$-3-5=-8$



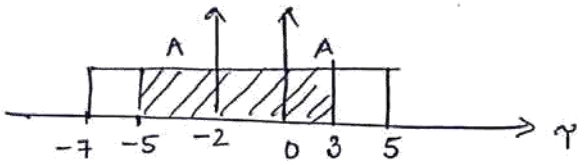
$$\int_{-5}^{2} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{2}$$

$$\Rightarrow A^2[2+5] = 7A^2.$$

⑩ at $t=-2$

$-2+5=+3$

$-2-5=-7$

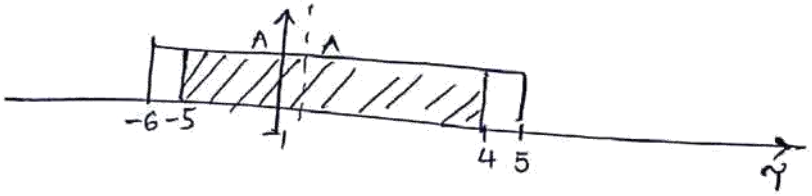


$$\int_{-5}^{3} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{3} = A^2[3+5] = 8A^2$$

⑪ at $t=-1$

$-1+5=4$

$-1-5=-6$



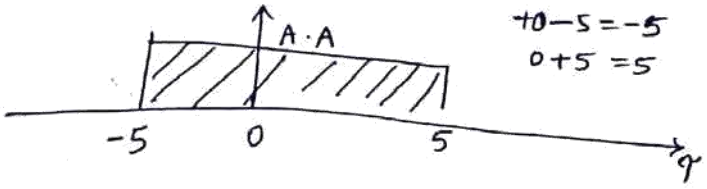
$$\Rightarrow \int_{-5}^{4} A\cdot A\, d\gamma \Rightarrow A^2[\gamma]_{-5}^{-4} = A^2[q]$$

$$= 9A^2$$

⑫ at $t=0$

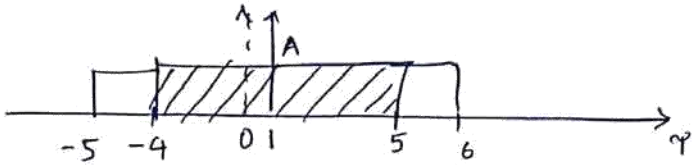
$0-5=-5$

$0+5=5$



$$\Rightarrow \int_{-5}^{5} A\cdot A\, d\gamma = A^2[\gamma]_{-5}^{5} = 10A^2$$

(13) at $t=1$



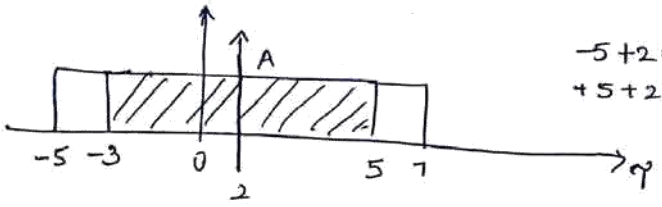
$-5+1=-4$

$+5+1=6$

$$\int_{-4}^{5} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-4}^{5}$$

$$\Rightarrow A^2[5+4] = 9A^2.$$

(14) at $t=2$



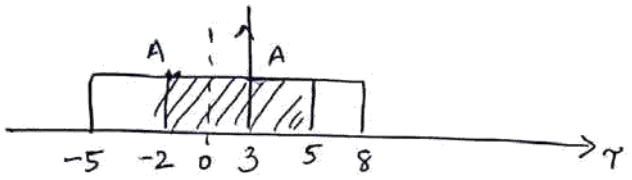
$-5+2=-3$

$+5+2=7$

$$\int_{-3}^{5} A\cdot A\, d\tau = A^2[\tau]_{-3}^{5}$$

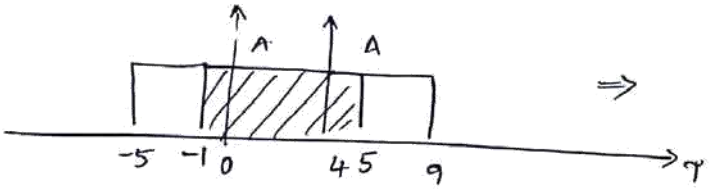
$$= 8A^2$$

(15) at $t=3$



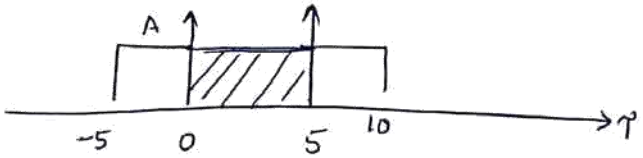
$$\int_{-2}^{5} A\cdot A\, d\tau \Rightarrow A^2[5+2] = 7A^2.$$

(16) $t=4$



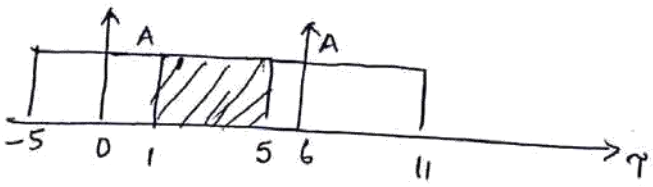
$$\Rightarrow \int_{-1}^{5} A\cdot A\, d\tau = A^2[5+1] = 6A^2$$

(17) $t=5$



$$\int_{0}^{5} A\cdot A\, d\tau = A^2[5]$$

(18) $t=6$



$$\int_{1}^{5} A\cdot A\, d\tau \Rightarrow A^2[\tau]^{5} = A^2[4]$$

$\|^{ly}$ for $t=7 \Rightarrow 3A^{\gamma}$

$t=8 \Rightarrow 2A^{\gamma}$

$t=9 \Rightarrow 1A^{\gamma}$

$t=10 \Rightarrow 0$

$t=-10 ; 10 = 0$

$t=-9 ; 9 = 1A^{\gamma}$

$t=-8 ; 8 = 2A^{\gamma}$

$t=-7 ; 7 = 3A^{\gamma}$

$t=-6 ; 6 = 4A^{\gamma}$

$t=-5 ; 5 = 5A^{\gamma}$

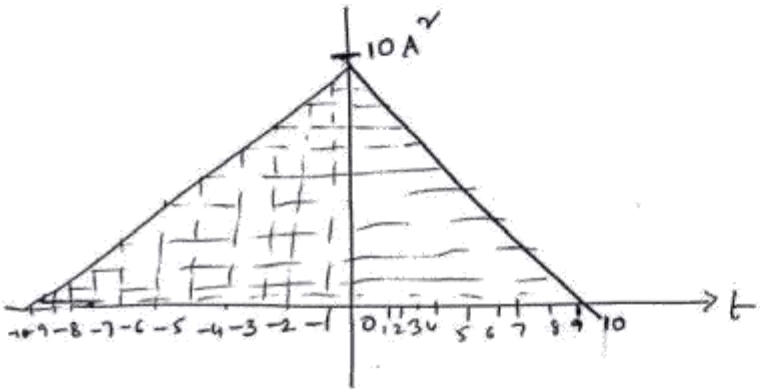
$t=-4 ; 4 = 6A^{\gamma}$

$t=-3 ; 3 = 7A^{\gamma}$

$t=-2 ; 2 = 8A^{\gamma}$

$t=-1 ; 1 = 9A^{\gamma}$

$t=0 \rightarrow 10A^{\gamma}$



Convolution: Mathematical way of combining two signals to form a third signal. i.e input signal, output signal and impulse response.

→ It is used to express the input and output relationship of a LTI system

→ If two functions $x(t)$ and $y(t)$ in time domain are defined then convolution $z(t)$ is

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau.$$

(OR)

$x(t) \otimes y(t)$.

↳ Read as $x(t)$ convolved with $y(t)$.

Convolution in time domain:

→ The convolution in time domain is equivalent to multiplication of their spectra in frequency domain i.e $x(t) \xleftrightarrow{X(\omega)}$ and $y(t) \longleftrightarrow Y(\omega)$ then

$$x(t) * y(t) \longleftrightarrow X(\omega) \cdot Y(\omega).$$

proof $x(t)$ fourier transform is given by

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$\text{But } x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\therefore F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Inter changing the order of integration, we get

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Using time shifting property, we get

$$\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt = Y(\omega)\, e^{-j\omega\tau}$$

$$F[x(t) * y(t)] = \int_{-\infty}^{\infty} x(\tau) Y(\omega) e^{-j\omega\tau} d\tau$$

$$= Y(\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = Y(\omega) X(\omega).$$

$$x(t) * y(t) \longleftrightarrow X(\omega) Y(\omega)$$

↓

This is time convolution theorem.

Convolution in frequency domain:

→ Multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain.

$x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then $2\pi x(t) y(t) \longleftrightarrow X(\omega) * Y(\omega)$ (or)

$x(t) y(t) \longleftrightarrow X(f) * Y(f)$.

proof

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$F\{x(t) y(t)\} = \int_{-\infty}^{\infty} x(t) y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda \right] y(t) e^{-j\omega t} dt$$

Interchanging the order of integration we get

$$F\{x(t) y(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \left[\int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} e^{j\lambda t} dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \left[\int_{-\infty}^{\infty} y(t) \cdot e^{-j(\omega-\lambda)t} dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) Y(\omega-\lambda) d\lambda. = \frac{1}{2\pi} [X(\omega) * Y(\omega)]$$

$$x(t) y(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$2\pi x(t) y(t) \longleftrightarrow X(\omega) * Y(\omega)$$

Graphical convolution:

→ The convolution by inspection provides the information needed without complicated calculations. This convolution by inspection procedure is called graphical convolution.

Procedure:

(1) $x(\tau)$ is the first function, where an independent variable 't' is replaced by dummy variable 'τ'.

(2) $y(-\tau)$ is the mirror image of $y(\tau)$ i.e $y(\tau)$ is flipped.

(3) $y(t-\tau)$ represents the function $y(-\tau)$ shifted to right by t seconds.

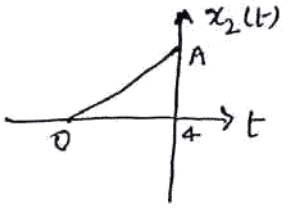
(4) For a particular value of $t=a$, integration of product $x(\tau)\,y(a-\tau)$ represents the area under the (common area) product curve.

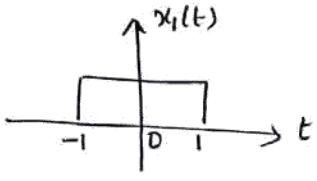
$$\int_{-\infty}^{\infty} x(\tau)\,y(a-\tau)\,d\tau = [x(t) * y(t)]_{t=a}$$

(5) The procedure is repeated for different values of t.
For negative value of t, the function $y(-\tau)$ is shifted left by t seconds.

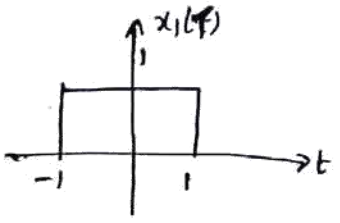
6) The value of convolution obtained at different values of t (i.e +ve, -ve).

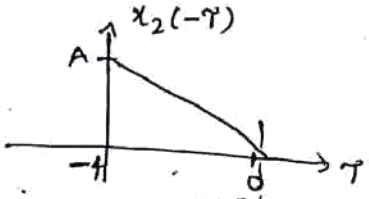
Q. Find the convolution of the functions $x_1(t)$ and $x_2(t)$

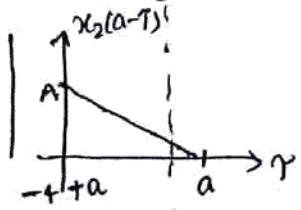




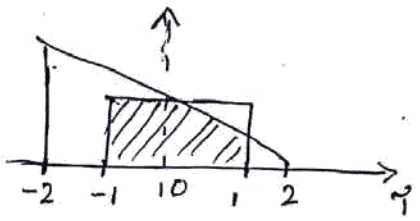
Sol)



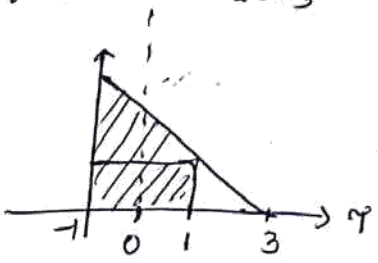




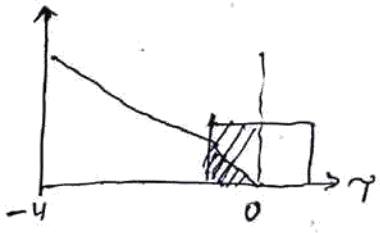
$[x_1(t) * x_2(t)]_{a=2}$

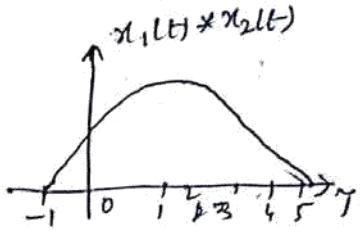


$[x_1(t) * x_2(t)]_{a=3}$



$[x_1(t) * x_2(t)]_{a=0}$





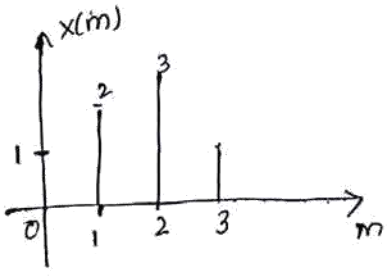
2) Determine the response of LTI system whose input $x(n)$ and impulse response $h(n)$ are given by $x(n) = \{1, 2, 3, 1\}$ and $h(n) = \{1, 2, 1, -1\}$

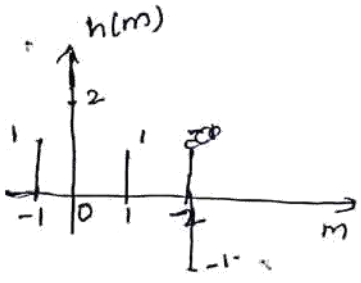
↑ ↑

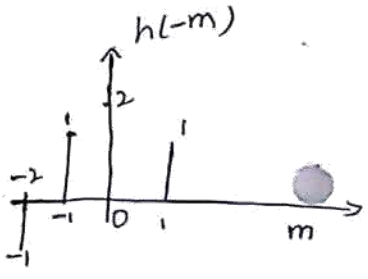
Sol The input sequence starts at $n=0$ and impulse response starts at $n=-1$.

O/p sequence starts at $n = 0 + (-1) = -1$

→ i/p & impulse response consists of 4 samples, so o/p consists of $4+4-1=7$ samples

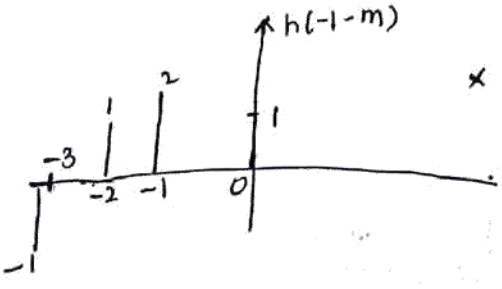




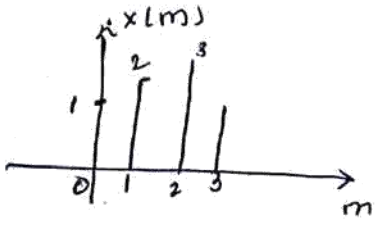


$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m).$$

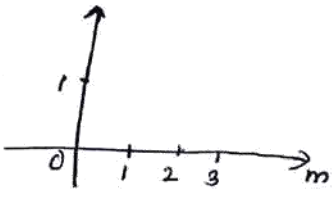
When $n=-1$; $y(-1) = \sum_{m=-\infty}^{\infty} x(m)\, h(-1-m)$



×



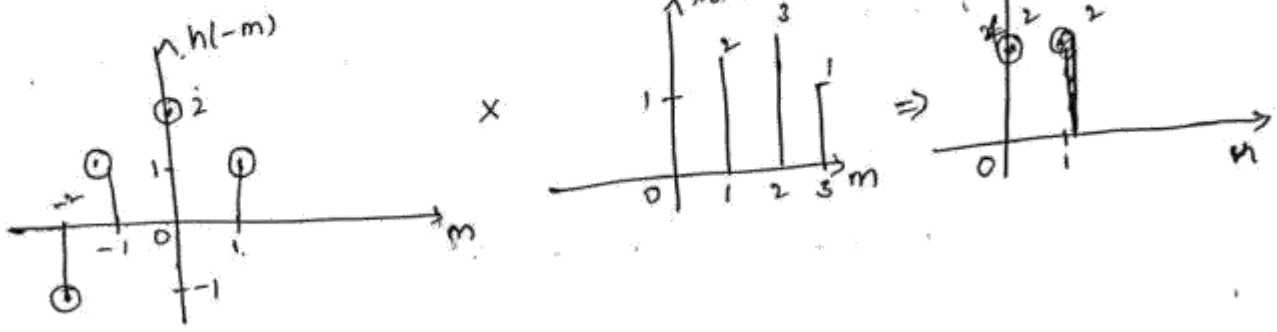
⇒



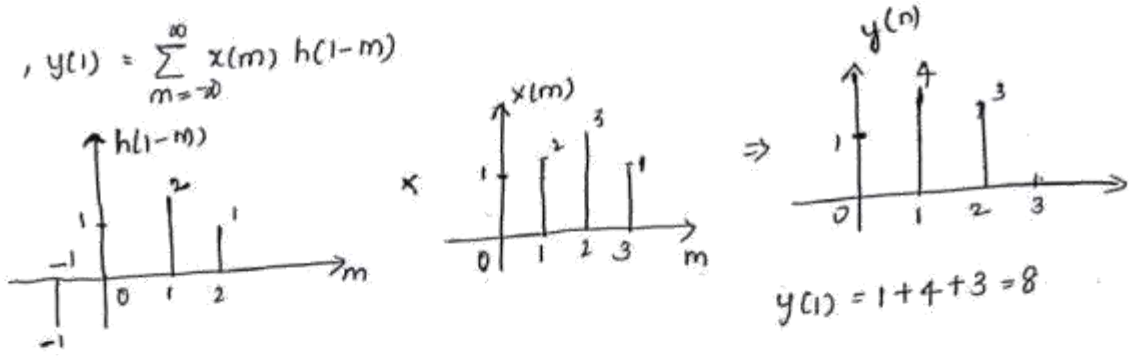
$y(-1) = 1$

When $n=0$, $y(0) = \sum_{m=-\infty}^{\infty} x(m)\, h(0-m)$.

$y(0) = 2+2 = 4$

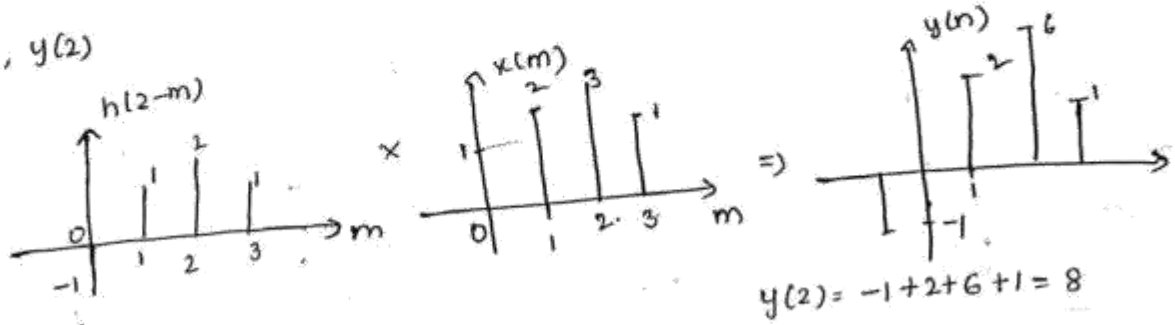


When $n=1$, $y(1) = \sum_{m=-\infty}^{\infty} x(m)\, h(1-m)$



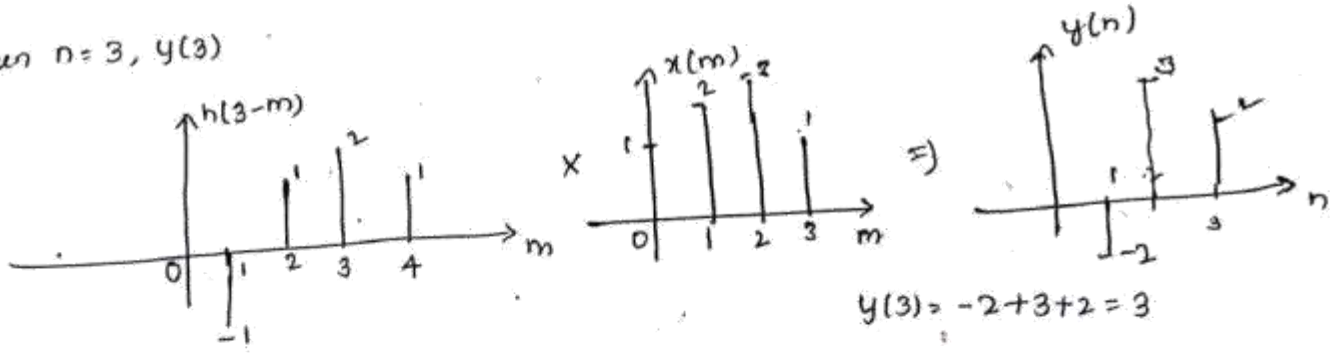
$y(1) = 1+4+3 = 8$

When $n=2$, $y(2)$



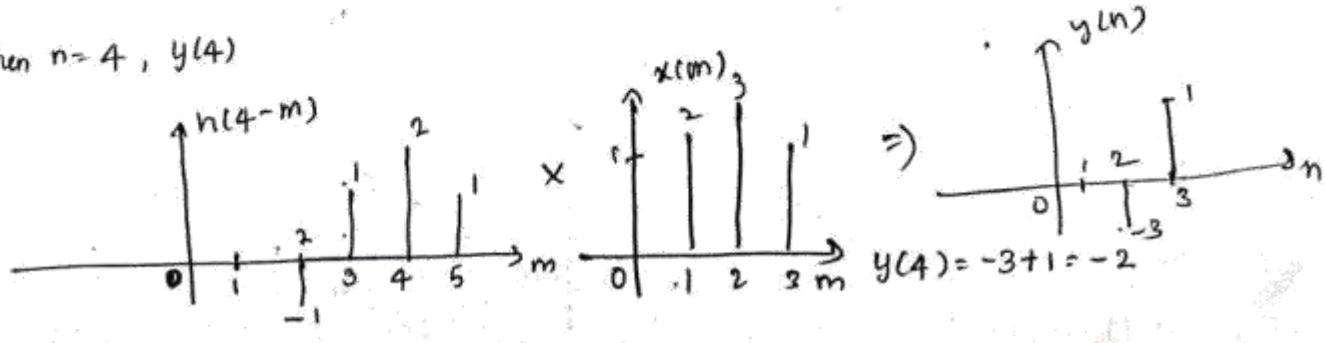
$y(2) = -1+2+6+1 = 8$

When $n=3$, $y(3)$



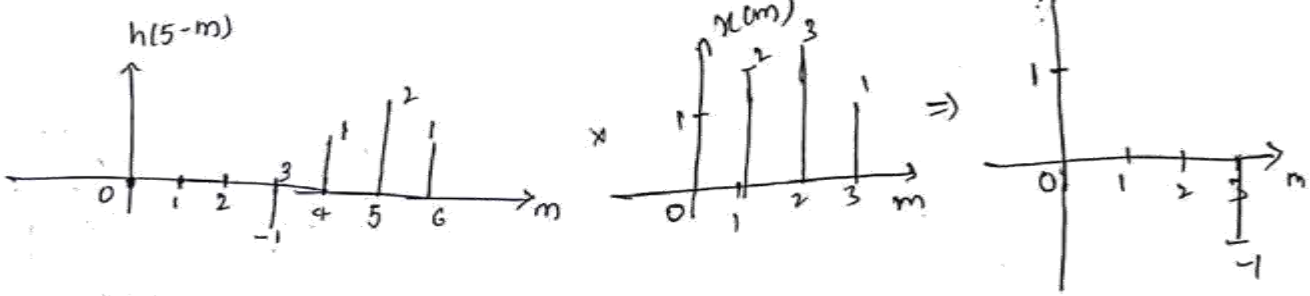
$y(3) = -2+3+2 = 3$

When $n=4$, $y(4)$



$y(4) = -3+1 = -2$

When n = 5

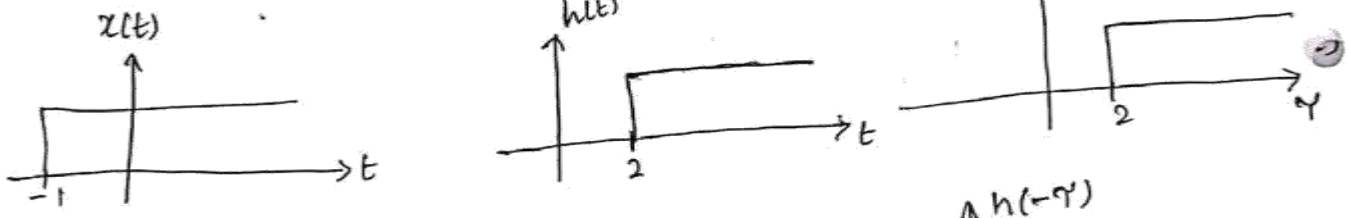


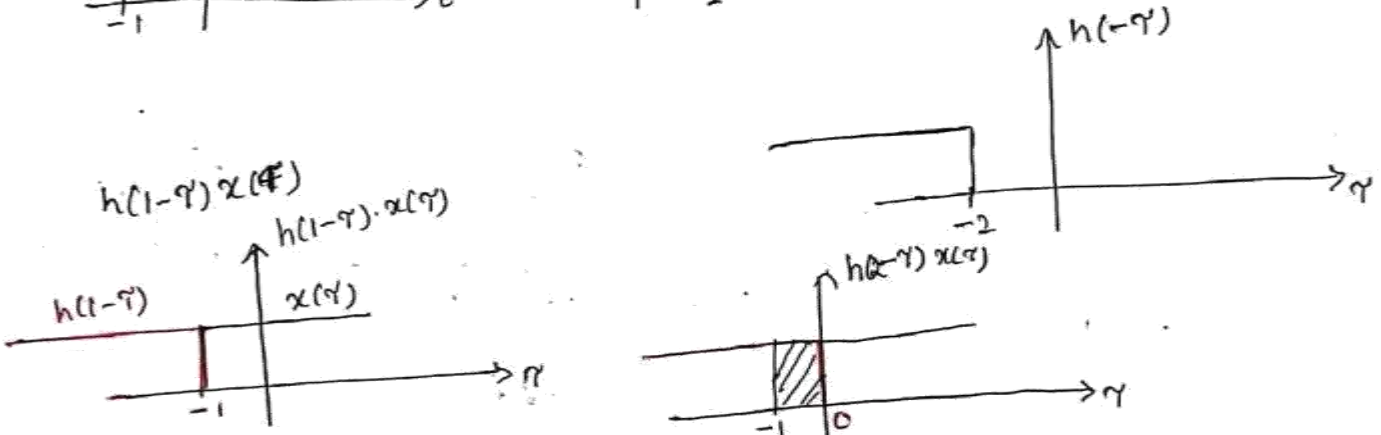
$y(5) = -1$

Output sequence $y(n) = \{1, 4, 8, 8, 3, -2, -1\}$

(3) Find the convolution of $x(t) = u(t+1)$ and $h(t) = u(t-2)$

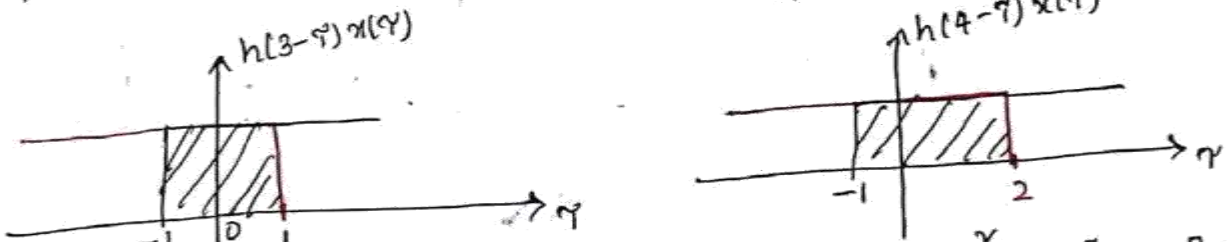
Sol





$y(1) =$ no overlap $= 0$

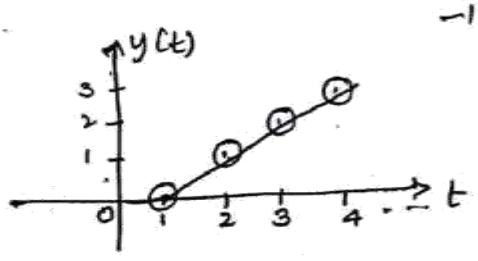
$y(2) = \int_{-1}^{0} 1 \cdot d\tau = [0-(-1)] = 1$



$y(3) = \int_{-1}^{1} 1 \cdot d\tau = [1-(-1)] = 2$

$y(4) = \int_{-1}^{2} 1 \cdot d\tau = [2-(-1)] = 3$

$\therefore y(t) =$



Condition

$y(t) = 0$ for $t \leq 0$

$y(t) = t-1$ for $t > 0$

(11) Find the convolution of $x(t)$ and $h(t)$

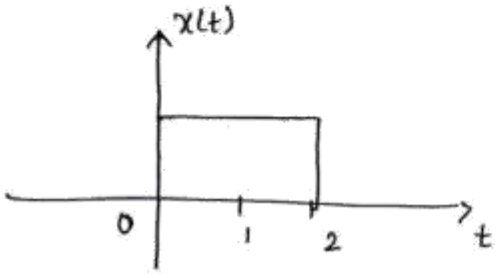
$x(t) = 1 \quad 0 \le t < 2$

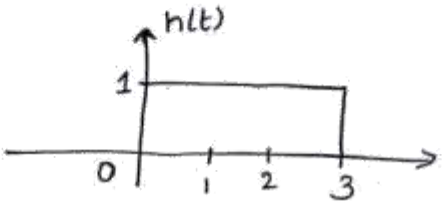
$= 0 \quad$ otherwise

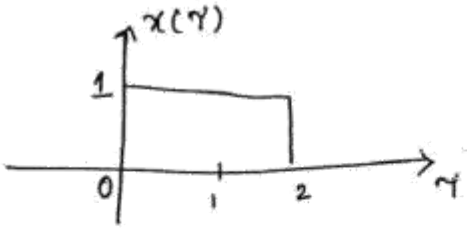
$h(t) = 1 \quad 0 \le t \le 3$

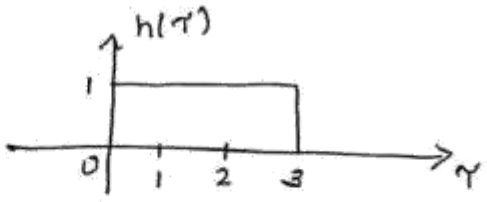
$= 0 \quad$ otherwise

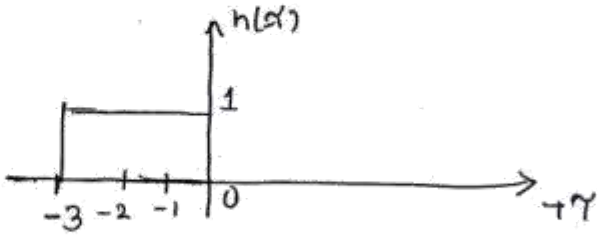
Sol



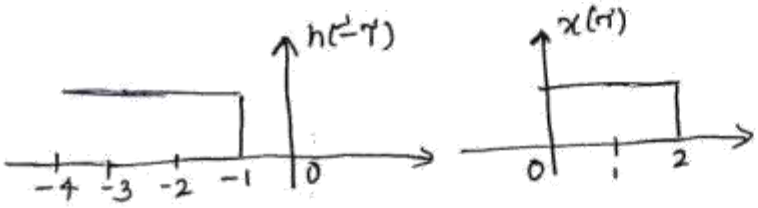






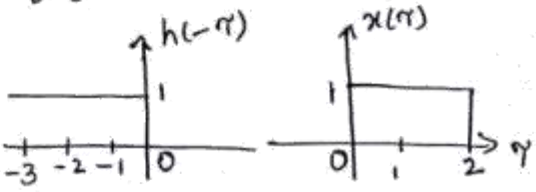


$t = -1$



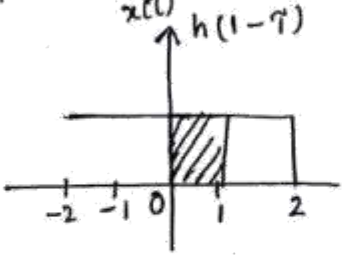
No overlap $\cdot\ y(t) = 0$

$t = 0$



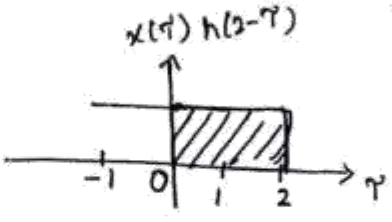
$y(t) = 0 \quad$ for $t \le 0$

$t = 1$



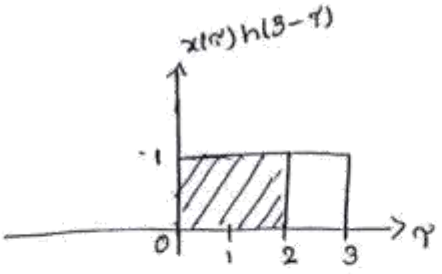
for $0 < t \le 1$

$$y(1) = \int_0^1 1 \cdot d\tau = [\tau]_0^1 = 1$$

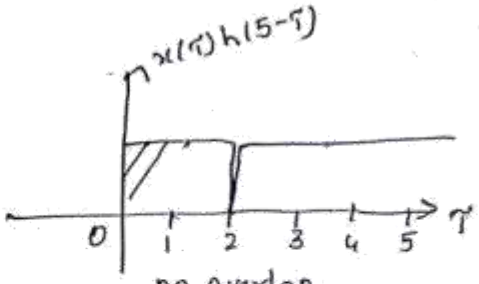


for $0 < t \le 2$

$$y(2) = \int_0^2 1 \cdot d\tau = [\tau]_0^2 = 2.$$

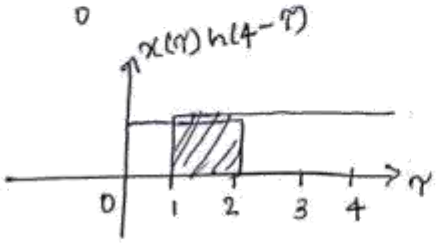


$$y(3) = \int_0^2 d\tau = 2$$

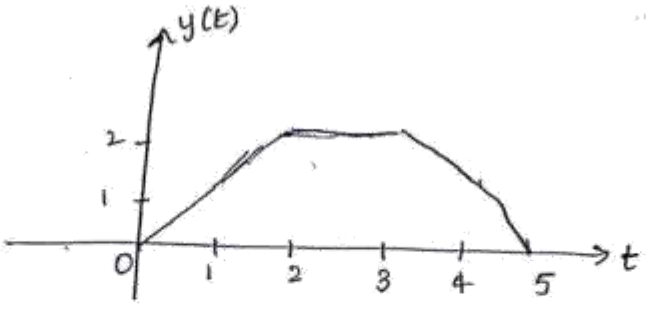


no overlap.

$y(5) = 0$.



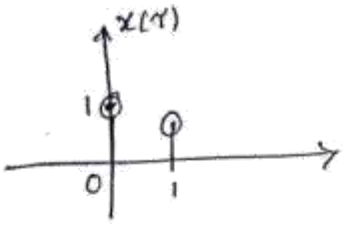
$$y(4) = \int_1^2 d\tau = [2-1] = 1$$

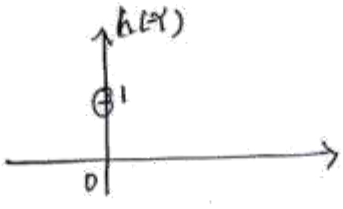


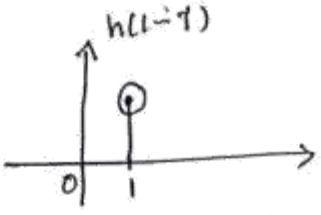
→ (5) Find the linear convolution $x(n) = \{1, 0.5\}$, $h(n) = \{1\}$

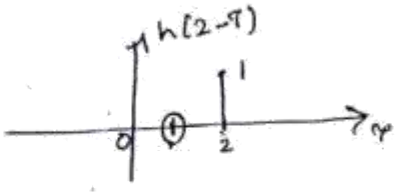
Sol $n_1 = 0$, $n_2 = 0$, so sequence starts at 0

$N_1 = 2$, $N_2 = 1$, $N_1 + N_2 - 1 = 2 + 1 - 1 = 2$





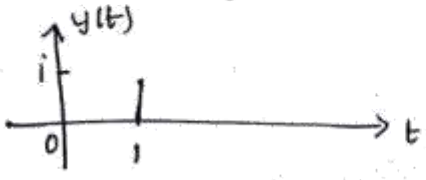




$y(0) = 1$

$y(1) = 0 + 0.5 \times 1 = 0.5$

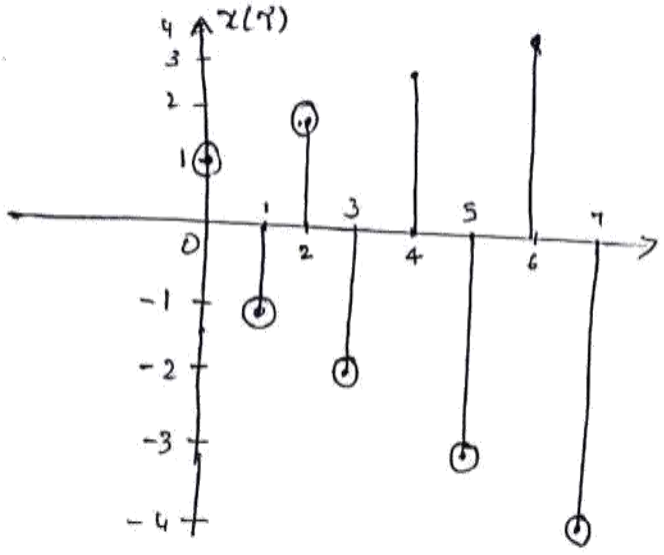
$y(2) = 0$.

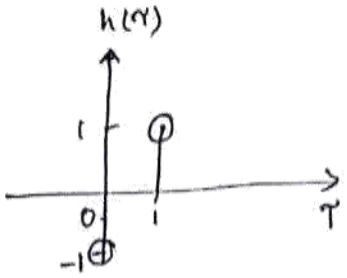


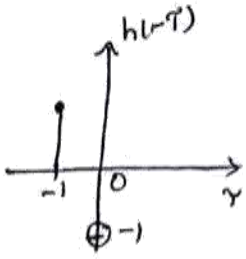
(8)

(6) Perform the linear convolution of $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$ $h(n) = \{-1, 1\}$

Sol =





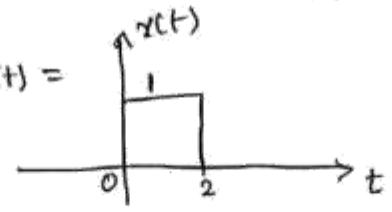


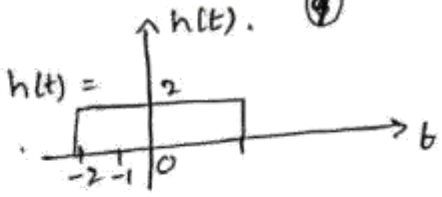
$n_1 = 0$, $n_2 = 0$ O/p sequence starts from 0

$N_1 = 8$, $N_2 = 2$, $N_1 + N_2 - 1 = 8 + 2 - 1 = 9$

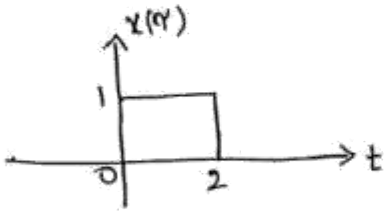
$y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, 4\}$

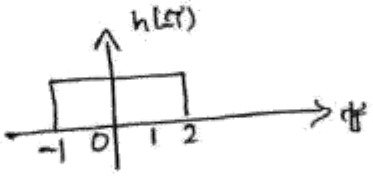
(7) Find the convolution of $x(t) =$



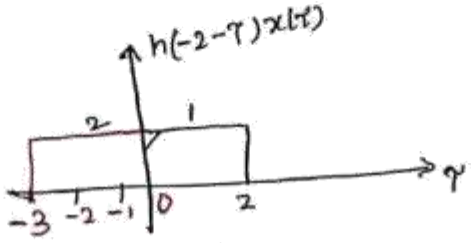


Sol



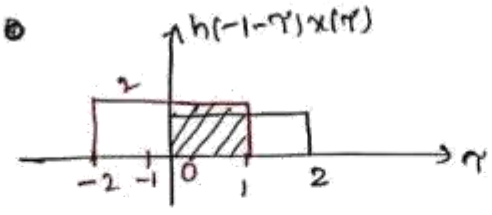


$y(-2) \Rightarrow 0$



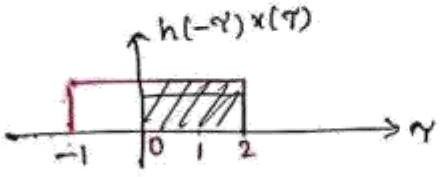
no overlap.

$y(-1) \Rightarrow$



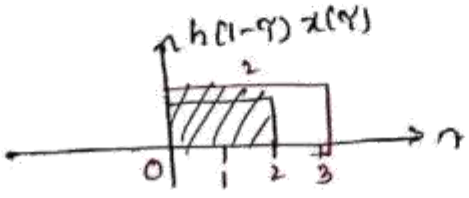
$$\Rightarrow \int_0^1 2.1\, d\tau = 2.$$

$y(0)$



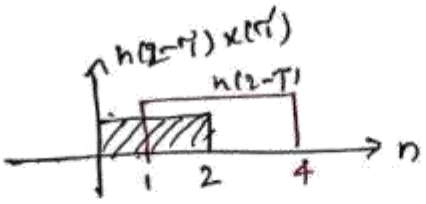
$$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$$

$y(1) =$



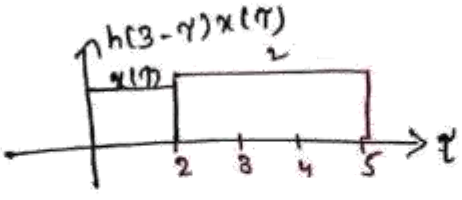
$$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$$

$y(2) =$

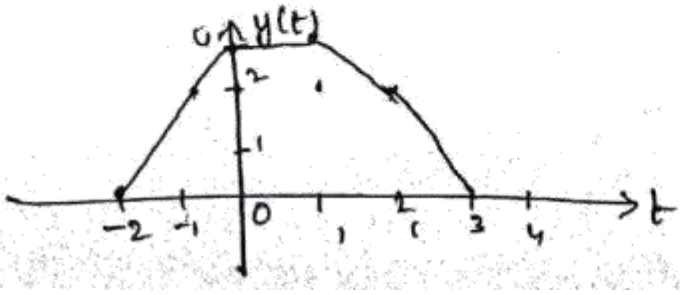


$$\Rightarrow \int_1^2 2.1\, dt = 2[2-1] = 2$$

$y(3) =$



$$= \int \text{no overlap} = 0$$



PROPERTIES OF CONVOLUTION:

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals is given by eqration.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

1. Commutative Property:

Convolution obeys commutative property.

$$x_1(t) * x_2(t) = x_2(t) * x_1(t).$$

Proof

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \rightarrow ①$$

Let $t-\tau = p \Rightarrow \tau = t-p$ | when $\tau = -\infty$, $p = t+\infty = \infty$

$-d\tau = dp$ | when $\tau = \infty$, $p = t-\infty = -\infty$

Substituting in eq① we get

$$x_1(t) * x_2(t) = -\int_{+\infty}^{-\infty} x_1(t-p)\, x_2(p)\, dp$$

$$= \int_{-\infty}^{\infty} x_2(p)\, x_1(t-p)\, dp$$

$$= x_2(t) * x_1(t)$$

2. Distributive Property:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t).$$

Proof

$$= x_1(t) * [x_2(t) + x_3(t)]$$

{considering $x_4(t) = x_2(t) + x_3(t)$}

$$= x_1(t) * x_4(t)$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_4(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, [x_2(t-\tau) + x_3(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x_1(\tau)\, x_3(t-\tau)\, d\tau$$

$$= [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)] \quad \text{RHS.}$$

(3) Associative property:

$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$ / $x_1(n) * [x_2(n) * x_3(n)] = [x_1(n) * x_2(n)] * x_3(n)$

Proof

Let $y_1(t) = x_1(t) * x_2(t)$

Let us replace by 'p'

$y_1(p) = x_1(p) * x_2(p)$

$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)$

Let $y_2(b) = x_2(b) * x_3(t)$

$y_2(b) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(b-q)$

$y_2(t-m) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(t-m-q)$

where p, m, and q are dummy variables.

RHS:

$[x_1(t) * x_2(t)] * x_3(t)$

$= y_1(t) * x_3(b)$

$= \sum_{p=-\infty}^{\infty} y_1(p)$

Proof

Let $y_1(n) = x_1(n) * x_2(n)$

Let us replace n by p

$y_1(p) = x_1(p) * x_2(p)$

$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)$

Let $y_2(n) = x_2(n) * x_3(n)$

$\therefore y_2(n) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(n-q)$

$\therefore y_2(n-m) = \sum_{q=-\infty}^{\infty} x_1(q)\, x_2(n-q-m)$

LHS:

$[x_1(n) * x_2(n)] * x_3(n)$

$= y_1(n) * x_3(n)$

$= \sum_{p=-\infty}^{\infty} y_1(p)\, x_3(n-p)$

$= \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(p-m)\, x_3(n-p)$

$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{p=-\infty}^{\infty} x_2(p-m)\, x_3(n-p).$

Let $p-m=q$

$\therefore p = m+q$

when $p=-\infty$, $q = p-m = -\infty-m = -\infty$

when $p=+\infty$ $q = p-m = +\infty-m = +\infty$

On replacing (p-m) by 'q' and 'p' by q+m.

$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{q=-\infty}^{\infty} x_2(q)\, x_3(n-q-m)$

$= \sum_{m=-\infty}^{\infty} x_1(m)\, y_2(n-m)$

$= x_1(n) * y_2(n)$

$= x_1(n) * [x_2(n) * x_3(n)]$

$=$ RHS

(4) Shift property:

If $x_1(t) * x_2(t) = z(t)$ then $x_1(t) * x_2(t-T) = z(t-T)$

Proof

$$x_1(t) * x_2(t-T) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-T-\tau)\, d\tau$$

$$= z(t-T)$$

Similarly $x_1(t-T) * x_2(t) = z(t-T)$ and $x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$.

(5) Convolution with impulse:

Convolution of a signal $x(t)$ with unit impulse is the signal $x(t)$ itself.

$$x(t) * \delta(t) = x(t).$$

Proof

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x(t)$$

$\left\{ \begin{array}{l} \delta(t-\tau) = 1 \text{ for } t = \tau \\ \quad\;\; = 0 \text{ otherwise} \end{array} \right.$

(6) Convolution with shifted impulse:

Convolution of a signal $x(t)$ with shifted impulse $\delta(t-t_0)$ is equal to $x(t-t_0)$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Proof

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau-t_0)\, d\tau$$

$$= x(\tau)\big|_{\tau = t-t_0} = x(t-t_0).$$

(7) Convolution with unit step:

Convolution of a signal $x(t)$ with unit step signal $u(t)$ is given by

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Proof

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau \qquad \because u(t-\tau) = 1 \text{ for } \tau < t \Rightarrow 0 \text{ for } \tau > t$$

Correlations:

→ Correlation is basically used to compare two signals or it is a measure of the degree to which two signals are similar.

→ Two types

(1) Cross-correlation

(2) Auto-correlation.

Cross Correlation:

The cross correlation between a pair of signals (sequences) $x_1(t)$ and $y(t)$ is given by

$$\gamma_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt \quad / \quad \gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \quad \text{where } l = 0, \pm 1, \pm 2, \cdots \text{ vi}$$

↳ ①

$$= \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\, dt$$

shift lag parameter.

→ The subscript xy indicates that $x(n)$ is the reference sequence that remains unshifted in time and $y(n)$ is shifted 'l' units in time w.r.to $x(n)$.

→ If we want to fix $y(n)$ and shift $x(n)$ then

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \quad \rightarrow ②$$

→ If time shift $l = 0$ then we get

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n).$$

Comparing eq(1) with eq(2) we find that

$$\gamma_{xy}(l) = \gamma_{yx}(-l)$$

↳ folded version of $\gamma_{xy}(l)$ abt $l=0$

We can rewrite the eq(1) as

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y[-(l-n)]$$

$$= x(l) * y(-l)$$

Auto Correlation :

It is the correlation of a sequence within itself.

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$$

(or)

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, x(n).$$

If time shift $l=0$, then

$$\gamma_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n).$$

Auto Correlation Signal :

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt$$

For real values

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt.$$

Properties of Cross Correlation function for energy signals.

1. For a real valued signals

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

Proof

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, y(t)\, dt$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t+\tau)\, x(t)\, dt = \int_{-\infty}^{\infty} y(t)\, x(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, y(t)\, dt = R_{xy}(-\tau)$$

For complex valued signals $R_{yx}(\tau) = R^*_{xy}(-\tau)$.

(2) If $R_{xy}(0) = 0$ then $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0$. Then the sgls are said to be orthogonal over the entire time interval.

Cross correlation of periodic signals:

The cross correlation between two periodic signals $x(t)$ and $y(t)$ is given by

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \quad \text{or} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t-\tau)\, dt.$$

$$\text{Similarly} \quad R_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, x^*(t-\tau)\, dt$$

Properties for periodic signals:

property 1: The fourier transform of cross correlation is equal to multiplication of fourier transform of one signal and complex conjugate of F.T of other signal

$$R_{xy}(\tau) \longleftrightarrow \frac{1}{T_0^2} \sum_{k=-\infty}^{\infty} X_1(kf_0)\, X_2^*(kf_0)\, \delta(f - kf_0).$$

2: If cross correlation is executed at origin ($\tau = 0$)

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t)\, dt = 0 \quad \text{i.e } R_{xy}(0) = 0. \text{ then sgls are said to be orthogonal}$$

3: Cross correlation exhibits conjugate symmetry

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

4. The cross correlation is not commutative

$$R_{xy}(\tau) \neq R_{yx}(\tau)$$

Properties for auto correlation function:

Property 1: The auto correlation is an even fn of τ. That is $R_x(\tau) = R_x(-\tau)$

If $x(t)$ is real valued → It has even symmetry

Otherwise

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$R^*_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

$$\therefore R^*_{xx}(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

$$= R_{xx}(\tau)$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt \rightarrow h_{xx}(\tau) \int_{-\infty}^{0} x(t)\, x(t+\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt \quad \Rightarrow \quad t+\tau = x \Rightarrow dt = dx.$$

$$= \int_{-\infty}^{0} f_1(x-\tau)\, f_1(x)\, dx$$

For complex → It has hermitian symmetry. changing the variable x to t, we

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt \qquad = \int_{-\infty}^{0} f_1(t-\tau)\, f_1(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt. \qquad = \phi_{11}(\tau)$$

i.e $R_{xx}(-\tau) = R^*_{xx}(\tau)$.

Property 2:

The auto correlation fn is bounded by its value at the origin That is

$$R_x(0) \geq |R_x(\tau)| \text{ for any } \tau$$

→ The largest value occurs at $\tau = 0$ of auto correlation fn.

proof Consider a finite energy signal $x(t)$.

Other method:

Consider the fn $x(t) \pm x(t+\tau)$

$$[x(t) + x(t+\tau)]^2 \geq 0$$

$$x^2(t) + x^2(t+\tau) \pm 2x(t)x(t+\tau) \geq 0$$

$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)x(t+\tau)$$

Integrating on b.s.

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt \geq 2\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$$\therefore E + E \geq 2R_{xx}(\tau)$$

$$E \geq R_{xx}(\tau)$$

$$\text{or } R(0) \geq R_{xx}(\tau)$$

$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - a x(t)|^2 dt$ → finite value. Obviously $|y(t)| > 0$

$$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - a x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt + |a|^2 \int_{-\infty}^{\infty} |x(t)|^2 dt - 2|a| \left| \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt \right|$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt$$

$$y(t) \geq \left[1 + |a|^2\right] R_{xx}(0) - 2|a|\, |R_{xx}(\tau)|$$

For $|a| = 1$

$$2R_{xx}(0) \geq 2|R_x(\tau)| \quad \text{since } |y(t)| > 0.$$

$$\Rightarrow R_x(0) > R_x(\tau)$$

Property 3:

The value of auto correlation fn at $\tau=0$ is equal to energy of the signal

$$E = R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Proof

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t)\cdot x^*(t)\, dt \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

Autocorrelation of power signals:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, x^*(t)\, dt$$

by putting $\tau=0$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\boxed{R_{xx}(0) = P}$$

∴ The autocorrelation fn at origin is equal to average power of the signal.

For periodic signals:

(1) If $R_{xx}(\tau)$ and $R^*_{xx}(-\tau)$ are complex conjugate of each other then $R(\tau) = R^*(-\tau)$ called as conjugate property.

Proof

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$R^*_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t-\tau)\, dt$$

$$R^*_{xx}(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t+\tau)\, dt = R(\tau)$$

Correlation:

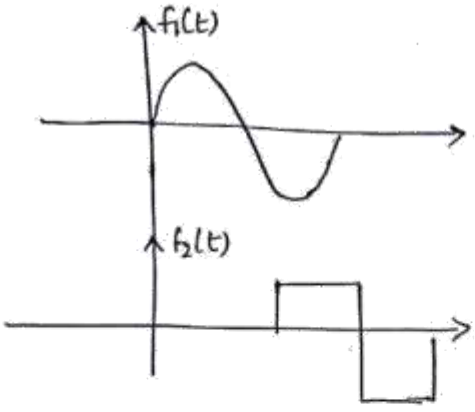
→ correlation means comparison

There are two types of correlations

(1) cross correlation

(2) Auto correlation.

Cross correlation: It is the measure of similarity between one waveform and time delayed version of the another waveform.

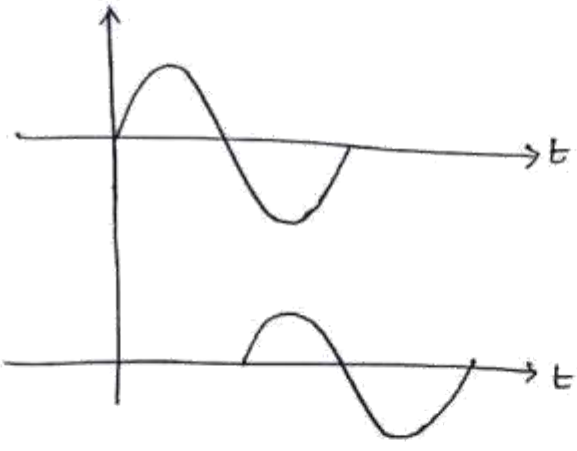


Equation of cross correlation is

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\, dt$$

Auto correlation: It is the measure of similarity between one waveform and time delayed version of the same waveform.

Ex:



Equation of auto correlation is

$$R_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_1^*(t-\tau)\, dt$$

APPLICATION OF CORRELATION:

→ It is used in the determination of signal that are contaminated with noise.

CROSS CORRELATION FOR ENERGY SIGNALS:

$$E = \int_{-\infty}^{\infty} f^2(t)\,dt \rightarrow \text{cross} \rightarrow \text{two fns}$$

Let $f_1(t)$ & $f_2(t)$ be two energy signals then cross co-relation b/w them is defined as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\,dt$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t+\tau)\, f_2^*(t)\,dt$$

Here τ is delay parameter or scanning parameter or scaling parameter

→ From the above two eqns cross correlation function obtained by shifting $f_2(t)$ in +ve direction by an amount τ is equal to the cross correlation function obtained by shifting $f_1(t)$ in -ve direction by an amount τ.

$$R_{21}(\tau) = \int_{-\infty}^{\infty} f_2(t)\, f_1^*(t-\tau)\,dt$$

$$R_{21}(\tau) = + \int_{-\infty}^{\infty} f_2(t+\tau)\, f_1^*(t)\,dt$$

By comparing with the convolution integral we can define cross correlation function of $f_1(t)$ & $f_2(t)$ as

$$R_{12}(\tau) = f_1(t) \otimes f_2(-t)$$

→ If $f_1(t)$ & $f_2(t)$ are even fns then cross correlation function becomes equal to convolution.

Differences between convolution & correlation

convolution	correlation
→ $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\,d\tau$	→ $R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\,dt$
τ = dummy variable	t = dummy variable.
→ convolution is a fn of physical time 't'	→ function of τ
→ It obeys commutation law $f_1(t) \otimes f_2(t) = f_2(t) \otimes f_1(t)$	→ It does not obey commutation law. $R_{12}(\tau) \neq R_{21}(\tau)$
→ It is used to evaluate the response of the system for arbitary i/p $f_1(t) \otimes f_2(t) = F_1(\omega)\, F_2(\omega)$	→ It is used to eliminate noise from the signals.

PROPERTIES OF CROSS CORRELATION:

Property 1: Proof: The F.T of cross correlation function of two signal is equal to product of FT of one signal and complex conjugate F.T of other signal.

$$F[R_{12}(\tau)] = X_1(f)\, X_2^*(f)$$

Proof

$$X_1(f) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi ft}\, dt \quad \rightarrow (1)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi ft}\, dt \quad \rightarrow (2)$$

put $t = t - \tau$

$dt = -d\tau$

$t = t - \tau$

$t = t + \infty = \infty$

$t = t - \infty = -\infty$

$+\infty = \infty + \tau$

$\frac{dt}{d\tau} = 0 - \frac{d\tau}{d\tau}$

$\frac{dt}{d\tau} = -1$

$\Rightarrow dt = -d\tau$

$$X_2(f) = \int_{+\infty}^{-\infty} x_2(t-\tau)\, e^{-j2\pi ft} \cdot e^{j2\pi f\tau}\, (-d\tau)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi ft} \cdot e^{j2\pi f\tau}\, d\tau$$

Applying conjugate on both sides

$$X_2^*(f) = \int_{-\infty}^{\infty} x_2^*(t-\tau)\, e^{j2\pi ft} \cdot e^{-j2\pi f\tau}\, d\tau \quad \rightarrow (3)$$

Now multiplying the eq(1) & (3) we get

$$X_1(f)\, X_2^*(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, e^{-j2\pi f\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{12}(\tau)\, e^{-j2\pi F\tau}\, d\tau \qquad \left[\because R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt\right]$$

$$\boxed{X_1(f)\, X_2^*(f) = F[R_{12}(\tau)]}$$

(2) The cross correlation fn is zero at origin i.e at $\tau = 0$,

$$R_{12}(\tau) = R_{12}(0) = 0$$

proof: Consider $R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$

$$R_{12}(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t)\, dt$$

$= 0$ ($\because x_1(t)$ & $x_2(t)$ are periodic).

3) The cross correlation fn satisfies conjugate symmetry property i.e $R_{12}(\tau) = R_{21}^*(-\tau)$

Proof

$$R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt \rightarrow (1)$$

$$R_{21}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_2(t)\, x_1^*(t-\tau)\, dt \rightarrow (2)$$

Putting $\tau = -\tau$

$$R_{21}(-\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_2(t)\, x_1^*(t+\tau)\, dt$$

Applying conjugate on both sides, we get

$$R_{21}^*(-\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_2^*(t)\, x_1(t+\tau)\, dt$$

{Now put substitute $t = t - \tau$.
$dt = dt$ }

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_2^*(t-\tau)\, x_1(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$= R_{12}(\tau)$$

$$\therefore R_{21}^*(-\tau) = R_{12}(\tau)$$

(4) The cross correlation fn does not satisfy commutative property i.e $R_{12}(\tau) \neq R_{21}(\tau)$

Proof

$$R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$R_{21}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_2(t)\, x_1^*(t-\tau)\, dt$$

As above eqn are not same

$$R_{12}(\tau) \neq R_{21}(\tau)$$

(5) The cross correlation for energy signals is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

(or)

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

is their cross

(8) Consider two energy signals $g_1(t)$ & $g_2(t)$, which may be complex valued. Show that $\int_{-\infty}^{\infty} |g_1(t) - g_2(t)|^2 dt = R_{11}(0) + R_{22}(0) - 2\,Re[R_{12}(0)]$

where $R_{11}(\lambda)$ and $R_{22}(\lambda)$ are autocorrelation fn of $g_1(t)$ & $g_2(t)$ & $R_{12}(\lambda)$

PROBLEMS ON AUTOCORRELATION AND CROSS CORRELATION

(1) Determine the cross correlation fn $R_{xy}(\lambda)$ $\rightarrow R_{12}(\lambda)$ of two signal $g_1(t)$ and $g_2(t)$ defined by

$$g_1(t) = \begin{cases} A\cos(2\pi f_1 t + \theta_1) & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \qquad g_2(t) = \begin{cases} A\cos(2\pi f_2 t + \theta_2) & 0 \le t < T \\ 0 & \text{elsewhere} \end{cases}$$

How does varying the frequency difference $|f_1 - f_2|$ affect this cross correlation fn.

Sol Cross correlation is given as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

or

$$R_{12}(\tau) = \int_{-\infty}^{\infty} g_1(t)\, g_2^*(t-\lambda)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2 (t-\lambda) + \theta_2)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2 t - 2\pi f_2 \lambda + \theta_2)\, dt$$

$$\left\{ \because \cos(A-B) + \cos(A+B) \right]$$

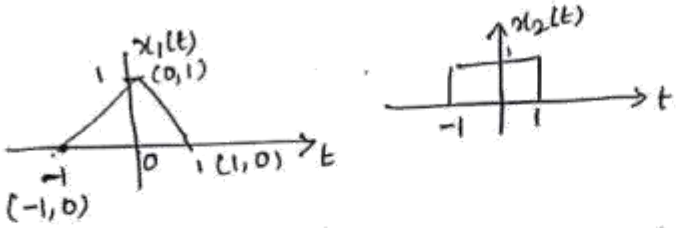
$$= \frac{A^2}{2} \int_0^T \Big\{ \cos(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda) + \cos(2\pi f_1 t + \theta_1 + 2\pi f_2 t - 2\pi f_2 \lambda + \theta_2) \Big\} dt$$

$$= \frac{A^2}{2} \left\{ \frac{\sin(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda)}{2\pi(f_1 - f_2)} \right]_0^T + \frac{\sin\left[2\pi(f_1+f_2)t + \theta_1 + \theta_2 - 2\pi f_2 \lambda\right]}{2\pi(f_1+f_2)} \Big]_0^T$$

$$= \frac{A^2}{2} \left\{ \frac{\sin(2\pi(f_1-f_2)T + \theta_1 - \theta_2 + 2\pi f_2 \lambda) - \sin[\theta_1 - \theta_2 + 2\pi f_2 \lambda)]}{2\pi(f_1 - f_2)} \right.$$

$$\left. + \frac{\sin\left[(2\pi(f_1+f_2)T + \theta_1 + \theta_2 - 2\pi f_2 \lambda) - \sin[\theta_1 + \theta_2 - 2\pi f_2 \lambda\right]}{2\pi(f_1+f_2)} \right]$$

2) Find the cross correlation b/w triangular and gate pulse as shown in fig.



Sol

$x_1(t)$ $(x_1, y_1) = (-1,0), (x_2, y_2) = (0,1)$

$y = x_1(t)$, $x = t$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$x_1(t) = t + 1$$

$$x_1(t) = \begin{cases} 1+t & \text{for } -1 \le t \le 0 \\ 1-t & \text{for } 0 \le t \le 1 \end{cases}$$

$x_2(t) = 1$ for $-1 \le t \le 1$

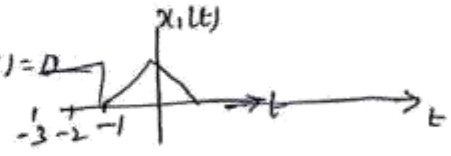
$x_2(t-\tau) = 1$ for $-1 \le t-\tau \le 1$

i.e $\tau - 1 \le t \le \tau + 1$

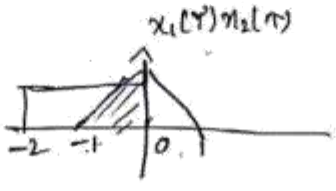
$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

Case i; $\tau < -2$

There is no overlap b/w $x_1(t)$ and $x_2(t)$ $\therefore R_{12}(\tau) = 0$



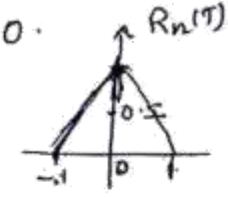
Case (ii) $-2 \le \tau \le -1$



$$R_{12}(\tau) = \int_{-1}^{0} (t+1)\, dt$$

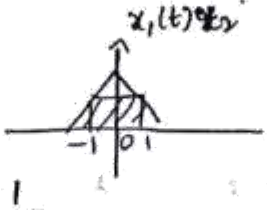
$$= \frac{1}{2} + 1 = \frac{3}{2}$$

$\frac{bh}{2} \Rightarrow \frac{1}{2} \times 1 \times 1 = 0.5$ between -1 to 0.



Case (iii)

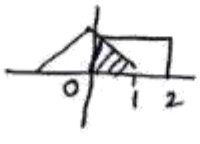
$$R_{12}(\tau) = \frac{1}{2} \times 1 \times 1 +$$



between -1 to 1

$$\frac{1}{2} \times 1 \times 1 = 1$$

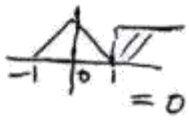
Case (iv) $= \frac{1}{2} \times 1 \times 1$



between 0 to 1

$= 0.5$

Case (v)



$= 0$.

Energy Density Spectrum:

Spectral density: It is the distribution of power or energy of a signal per unit bandwidth as a function of frequency.

Energy and power signals:

→ Signals with finite energy i.e $0<E<\infty$ and $P=0$ are called energy signals.
e.g: aperiodic signals like pulse

→ Signals with finite average power i.e $0<P<\infty$ and $E=\infty$ are called power sgls
i.e periodic signals.

The energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Parseval's Theorem for energy signal: / Rayleigh energy theorem:

It defines the energy of a signal in terms of fourier transform.

$$\text{i.e } E = \int_{-\infty}^{\infty} |x(f)|^2 df.$$

Proof $x(t) \longleftrightarrow x(f)$. Let $x^*(t)$ be conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow x^*(-f)$$

Energy of the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt.$$

Replacing $x(t)$ in terms of inverse fourier transform $x(f)$

$$E = \int_{-\infty}^{\infty} x^*(t) \left\{ \int_{-\infty}^{\infty} x(f)\, e^{j\omega t}\, df \right\} dt$$

Interchanging the order of integration.

$$E = \int_{-\infty}^{\infty} x(f) \left\{ \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt \right\} df$$

$$E = \int_{-\infty}^{\infty} x(t)\, x^*(f)\, df \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 df.$$

Energy Spectral density: It is the distribution of energy of signal in frequency domain. which is also called as energy density spectrum (ESD or ED) given by

$$\psi(f) = |X(f)|^2 \quad \rightarrow (1)$$

Let $x(t)$ and $y(t)$ be the input and output of a linear system. i.e $x(t) \leftrightarrow X(f)$ and $y(t) \leftrightarrow Y(f)$ and $H(f)$ be system transfer function.

$$Y(f) = H(f)X(f) \rightarrow (2)$$

Using Eq (1) we can write as

$$\psi_x(f) = |X(f)|^2$$

$$\psi_y(f) = |Y(f)|^2.$$

$$\psi_y(f) = |Y(f)|^2 = |H(f)|^2 |X(f)|^2 = |H(f)|^2 \psi_x(f).$$

$$\boxed{\psi_y(f) = |H(f)|^2 \psi_x(f)}$$

ESD of the output is the product of ESD of input and square of the magnitude of transfer function.

Power Density Spectrum:

Average power is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

But power P is defined as

$$P = \overline{x^2(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Parseval's Power Theorem: It defines the power of a signals in terms of its fourier series coefficients.

$$P = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Proof : Consider a function $x(t)$

$$|x(t)|^2 = x(t)\, x^*(t) \quad \rightarrow (1)$$

($x^*(t)$ → conjugate of $x(t)$.)

Average power of $x(t)$ for one cycle is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt \quad \rightarrow (2)$$

we have exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad \rightarrow (3)$$

Replacing $x(t)$ of eq(2) by eq(3), we get

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\, x^*(t)\, dt$$

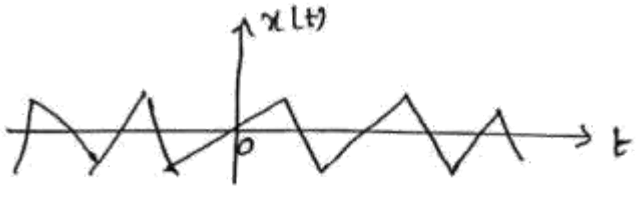
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t}\, dt = \sum_{n=-\infty}^{\infty} F_n \cdot \frac{1}{T}\int_{-T/2}^{T/2} x^*(t) e^{jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} F_n \cdot F_n^* = \sum_{n=-\infty}^{\infty} |F_n|^2 = P.$$

↳ parsevals power theorem.

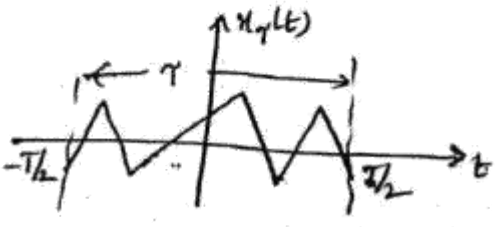
Power Spectral Density (PSD):

The distribution of average power of the signal in frequency domain is called power spectral density or power density spectrum (PSD or PD)



Let us assume that signal is zero outside the interval $(T/2)$.

$$x_T(t) = \begin{cases} x(t) & |t| < T/2 \\ 0 & \text{elsewhere} \end{cases}$$



The signal $x_T(t)$ is of finite duration T and hence it energy signal with energy E given by

$$E = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

where $x_T(t) \longleftrightarrow X_T(f)$.

As $x(t)$ over interval $(-T/2, T/2)$ is same as $x_T(t)$ over the interval $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\therefore \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 = \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2 df \rightarrow (1)$$

If $T \to \infty$, the left hand side of eq(1) represents average power P

$$P = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} df$$

If $T \to \infty$, $\frac{|X_T(f)|^2}{T}$ approaches finite value denoted by $S(f)$ or $S(\omega)$

$$S(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$

Average power

$$P = \overline{x^2(t)} = \int_{-\infty}^{\infty} S(f)\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega.$$

→ The PSD of periodic function is given by

$$S(f) = \sum_{n=-\infty}^{\infty} |Fn|^2 \delta(f - n f_0)$$

(or) by $S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |Fn|^2 \delta(\omega - n\omega_0)$.

I/p and o/p relation of linear system in terms of PSD is given by

$$S_y(f) = |H(f)|^2 \cdot S_x(f).$$

Comparsion of ESD & PSD

ESD	PSD
(1) It gives the distribution of energy of a signal in frequency	(1) It gives the distribution of power of signal in frequency domain
(2) It is given by $\psi(f) = \lvert X(f)\rvert^2$	(2) It is given by $S(f) = \lim_{T\to\infty} \frac{\lvert X_T(f)\rvert^2}{T}$
(3) Total energy is given by $E = \int_{-\infty}^{\infty} \psi(f)\,df$	(3) Total power is given by $P = \int_{-\infty}^{\infty} S(f)\,df$
(4) The autocorrelation for an energy signal and its ESD form a Fourier transform pair $R(\tau) \longleftrightarrow \psi(f)$	(4) The autocorrelation for a power signal and its PSD form a Fourier transform pair. $R(\tau) \longleftrightarrow S(f)$

RELATION BETWEEN AUTOCORRELATION AND SPECTRAL DENSITIES

1. The autocorrelation function $R(\tau)$ of an energy signal and its energy spectral density (ESD), $\psi(f)$ forms a fourier transform pair,

$$R(\tau) \longleftrightarrow \psi(f)$$

proof: Cross correlation of two energy signals $x(t)$ and $y(t)$ is given as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, e^{j\omega\tau}\, df.$$

If both functions are same, then auto correlation is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(f)\, X^*(f)\, e^{j\omega\tau} df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 e^{j\omega\tau} df$$

$$= F^{-1}\left[\lvert F(f)\rvert^2\right]$$

But $\lvert F(f)\rvert^2 = \psi(f)$

$$\therefore R(\tau) = F^{-1}[\psi(f)] \Rightarrow F[R(\tau)] = \psi(f)$$

$$R(\tau) \longleftrightarrow \psi(f)$$

2) The autocorrelation function $R(\tau)$ and power spectral density (PSD), $S(f)$ of a power signal form a fourier transform pair.

$$R(\tau) \longleftrightarrow S(f)$$

Proof

Autocorrelation function of power $x(t)$ in terms of fourier series coefficients is given as

$$R(\tau) = \sum_{n=-\infty}^{\infty} X_n X_{-n} e^{jn\omega_0\tau}$$

X_n : exponential fourier series coefficients

$$R(\tau) = \sum_{n=-\infty}^{\infty} |X_n|^2 e^{jn\omega_0\tau}$$

Taking Fourier transform.

$$F[R(\tau)] = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} |X_n|^2 e^{jn\omega_0\tau} \right] e^{-j\omega\tau} d\tau$$

Interchanging the order of integration & summation, we get

$$F[R(\tau)] = \sum_{n=-\infty}^{\infty} |X_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)} d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 \cdot \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0).$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of periodic function $x(t)$

$$\therefore F[R(\tau)] = S(f)$$

$$R(\tau) = F^{-1}[S(f)]$$

$$\therefore R(\tau) \longleftrightarrow S(f).$$

RELATION BETWEEN CONVOLUTION AND CORRELATION:

(1) In correlation, physical time 't' plays the role of dummy variable & it appears after solving the integral but in convolution delay parameter τ plays the role of dummy variable.

2) Correlation $R_{xy}(\tau)$ is a function of delay parameter τ, whereas convolution is a function of time.

3) Correlation can be obtained by convolving $x(t)$ & $y^*(t)$

4) Convolution does not depend on which function is being shifted whereas correlation does i.e convolution is commutative.

(7) The convolution and correlation are identical for even signals.

Proof For two signals $x_1(t)$ and $x_2(t)$

The definition for correlation is given by

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt \quad \rightarrow (1)$$

$$R_{21} = \int_{-\infty}^{\infty} x_2(t)\, x_1(t-\tau)\, dt \quad \rightarrow (2)$$

The definition for convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \quad \rightarrow (3)$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau \quad \rightarrow (4)$$

Taking eq (1) & eq (3)

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

Replacing dummy variable 't' by 'p' we get

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(p-\tau)\, dp$$

Since it is a even signal i.e $x(t) = x(-t)$

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(-(\tau-p))\, dp = S$$

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(\tau-p)\, dp$$

Replace dummy variable τ by t

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(t-p)\, dp$$

Again replacing the variable p by τ

$$S = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= x_1(t) * x_2(t).$$

Extension of Autocorrelation Property.

(4) Fourier transform of auto correlation yields π times the energy density spectrum of $f_1(t)$ $\quad |F_1(\omega)|^2 = \int_{-\infty}^{\infty} R_{11}(\tau) e^{-j\omega t} d\tau = \pi S_1(\omega)$

proof

From defination of FT, FT of $R_{11}(\tau)$ is

$$\int_{-\infty}^{\infty} R_{11}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t) \cdot f_1(t-\tau) e^{-j\omega\tau} \cdot dt\, d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt \int_{-\infty}^{\infty} f_1(t-\tau) e^{j\omega(t-\tau)} d\tau$$

putting $t-\tau = -x \Rightarrow d\tau = dx$

$$= F_1(\omega) \int_{-\infty}^{\infty} f_1(-x) e^{-j\omega x} dx$$

$$= F_1(\omega) F_1(-\omega)$$

$$= |F_1(\omega)|^2 = \pi S_1(\omega).$$

(5) Fourier transform of R_{12} (cross correlation fn) is $F_1(\omega) F_2(-\omega) \leftrightarrow F_1(\omega) F_2(-\omega)$

proof Fourier transform of $f_1(t)$ & $f_2(t)$ are

$$f_1(t) \leftrightarrow F_1(\omega),\ f_2(t) \leftrightarrow F_2(\omega) \text{ i.e. } f_1(-t) \leftrightarrow F_1(-\omega) \text{ \& } f_2(-t) \leftrightarrow F_2(-\omega)$$

$$R_{12}(\tau) = f_1(t) * f_2(-t).$$

$\therefore$ Fourier transform of $f_1(t) * f_2(-t)$ is $F_1(\omega) F_2(-\omega)$.

$\therefore$ $R_{12}(\tau) = f_1(t) * f_2(-t) \longleftrightarrow F_1(\omega) \cdot F_2(-\omega)$.

(6) Graphically $R_{12}(\tau)$ is same as $R_{21}(\tau)$ where it is folded back about the vertical axis at $\tau = 0$. $R_{12}(\tau) = R_{21}(-\tau)$.

proof

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) \cdot f_2(t-\tau)\, dt = \int_{-\infty}^{\infty} f_1(t+\tau) f_2(t)\, dt$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} f_2(t) \cdot f_1(t-\tau)\, dt$$

$$R_{21}(-\tau) = \int_{-\infty}^{\infty} f_2(t) \cdot f_1(t-(-\tau))\, dt = \int_{-\infty}^{\infty} f_2(t) f_1(t+\tau)\, dt = R_{12}(\tau)$$

ESD (Energy Spectral Density):

→ Gives the distribution of energy of the signal in the frequency domain.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega$$

we know

$$\omega = 2\pi f$$

$$d\omega = 2\pi df$$

$$\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(f)|^2 \cdot 2\pi df = \int_{-\infty}^{\infty} |x(f)|^2 df \quad \text{coz } |x(2\pi f)| \text{ is written as } x(f). \quad \hookrightarrow (1)$$

where $|x(f)|$ → amplitude spectrum.

If we denote $|x(f)|^2$ by $\psi(f)$

$$\therefore \text{ESD}: \psi(f) = |x(f)|^2 \rightarrow (2)$$

putting (2) in eq (1)

$$E = \int_{-\infty}^{\infty} \psi(f) df.$$

↓ total energy of the sgl.

→ total area under the curve $\psi(f)$.

→ $\psi(f)$ represents → Energy spectral density of sgl $x(t)$ in joules per hertz.

Effect of Systems on ESD

Let the ESD of $x(t)$ be $\psi_x(f)$ and $y(t)$ be $\psi_y(f)$. The signal $x(t)$ is applied at the input of LTI system and $y(t)$ is obtained at the output.

→ Let LTI system to be an ideal filter which has pass band from f_L to f_H. i.e only signal will be passed without any effect from f_L to f_H.

we know that $E = \int_{-\infty}^{\infty} \psi(f)\, df$

Energy at the output will be

$$E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df.$$

→ If $\psi_y(f)$ is symmetric for positive & negative values of 'f', then

$$E_y = 2\int_{0}^{\infty} \psi_y(f)\, df.$$

$$E_y = 2\int_{f_L}^{f_H} \psi_y(f)\, df$$

$$= 2\int_{f_L}^{f_H} |Y(f)|^2 df.$$

We know $Y(\omega) = H(\omega)\, X(\omega)$.

$$E_y = 2\int_{f_L}^{f_H} |H(f)X(f)|^2 df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2 |X(f)|^2 df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2 \psi_x(f)\, df$$

→ The filter passes all the frequencies b/w f_L & f_H i.e $H(f) = 1$ for $f_L \leq f \leq f_H$.

$$E_y = 2\int_{f_L}^{f_H} \psi_x(f)\, df$$

↙ energy of the o/p sgl. in terms of ESD of i/p sgl.

RELATION BETWEEN CORRELATION AND CONVOLUTION:

The convolution of $f_1(t)$ and $f_2(-t)$ by $\rho_{12}(t)$ we have

$$\rho_{12}(t) = f_1(t) * f_2(-t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t)\, d\tau$$

The dummy variable τ in the above integral may be replaced by another variable x.

$$\rho_{12}(t) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-t)\, dx.$$

changing the variable from t to τ, we get

$$\rho_{12}(t) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-\tau)\, dx$$

$$= \phi_{12}(\tau)$$

Hence $\phi_{12}(\tau) = f_1(t) * f_2(-t)\big|_{t=\tau} = \rho_{12}(\tau)$

Ily $\phi_{21}(\tau) = f_1(-t) * f_2(t)\big|_{t=\tau} = \rho_{21}(\tau).$

and $\phi_{11}(\tau) = f_1(t) * f_1(-t)\big|_{t=\tau} = \rho_{11}(\tau)$

DETECTION OF PERIODIC SIGNALS IN THE PRESENCE OF NOISE BY CORRELATION:

→ Now we consider that periodic signals are affected by noise that finds the applications in the detection of radar and sonar signals, periodic component in brain waves and cyclical component in ocean wave analyses.

→ If $s(t)$ is periodic signal and $n(t)$ represents the noise signal, then

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)\, n(t-\tau)\, dt = 0 \quad \text{for all } \tau.$$

$\phi_{sn}(\tau) = 0$

↙

cross correlation fn

Detection by Autocorrelation:

Let $s(t)$ be a periodic signal mixed with noise signal $n(t)$. Then the received sgl $f(t)$ is $[s(t)+n(t)]$.

→ Let $\bar{\phi}_{ff}(\tau)$, $\bar{\phi}_{ss}(\tau)$, $\bar{\phi}_{nn}(\tau)$ denote the autocorrelation functions of $f(t)$, $s(t)$ and $n(t)$ respectively.

$$\bar{\phi}_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, f(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [s(t)+n(t)]\,[s(t-\tau)+n(t-\tau)]\, dt$$

$$= \bar{\phi}_{ss}(\tau) + \bar{\phi}_{nn}(\tau) + \bar{\phi}_{sn}(\tau) + \bar{\phi}_{ns}(\tau)$$

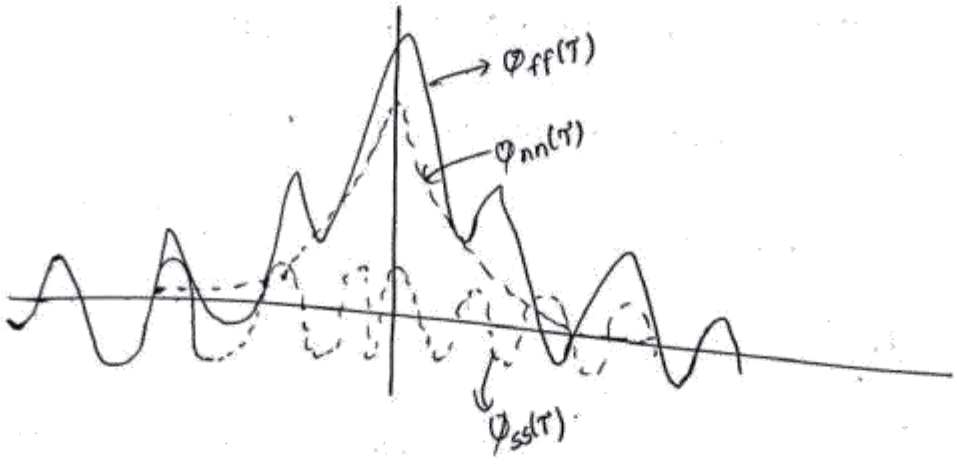
($\because$ $s(t)$ & $n(t)$ are uncorrelated)

$$\bar{\phi}_{sn}(\tau) = \bar{\phi}_{ns}(\tau) = 0$$

$$\therefore \quad \bar{\phi}_{ff}(\tau) = \bar{\phi}_{ss}(\tau) + \bar{\phi}_{nn}(\tau).$$

↓

exhibit a periodic nature at larger values of τ.



→ It follows that $f(t)$ contains a periodic signal of frequency displayed by $\bar{\phi}_{ff}(\tau)$.

→ If $\bar{\phi}_{ff}(\tau)$ does exhibit such a periodic nature it is possible to separate $\bar{\phi}_{ss}(\tau)$ and $\bar{\phi}_{nn}(\tau)$ {non periodic component}.

EXTRACTION OF SIGNAL FROM NOISE BY FILTERING:

→ A signal masked by noise can be detected either by correlation techniques or filtering.

→ Correlation technique in time domain and filtering in frequency domain.

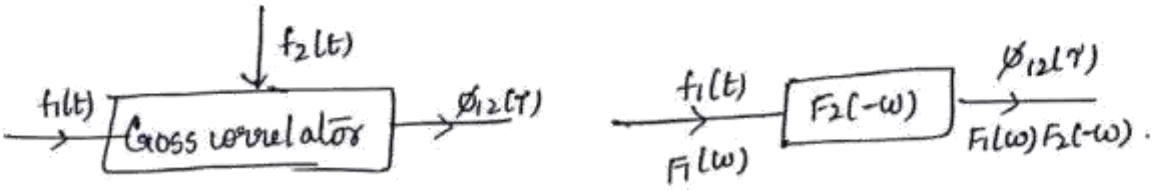


Fig: Crosscorrelation in time & frequency domain.

The impulse response $h(t)$ of a system with a transfer function $F_2(-\omega)$ is given by

$$h(t) = F^{-1}[F_2(-\omega)]$$

But $f_2(t) \longleftrightarrow F_2(\omega)$

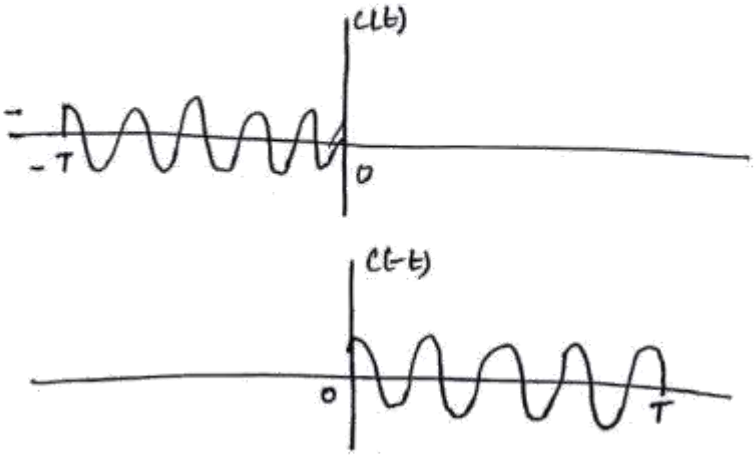
and $f_2(-t) \longleftrightarrow F_2(-\omega)$

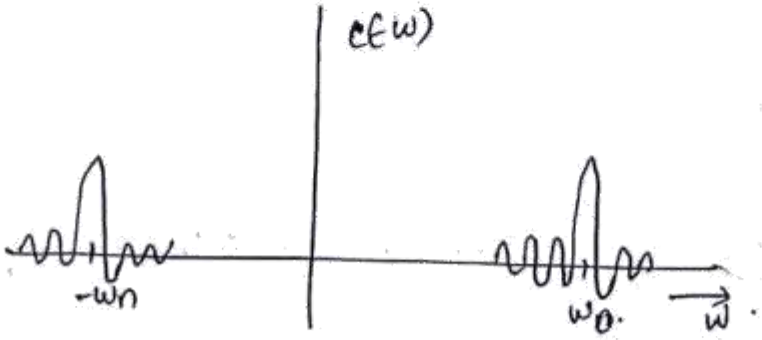
∴ hence $h(t) = f_2(-t)$.

→ The received signal $f(t)$ is

$$f(t) = s(t) + n(t)$$

→ We are filtering out all of the noise signal and extracting the desired periodic signal $s(t)$ by a filter which allows only the frequency components present in $s(t)$ to pass through.




E(w)
-wn
w0
w